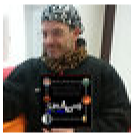
RISORSE DIDATTICHE.



【ℜĢ】 By … 0000-0001-5086-7401 & lnkd.in/erZ48tm

RISORSE DIDATTICHE
EDUCAZIONE CIVICA

Annals of Mathematics

The Discovery of Incommensurability by Hippasus of Metapontum
Author(s): Kurt Von Fritz
Source: *Annals of Mathematics*, Second Series, Vol. 46, No. 2 (Apr., 1945), pp. 242-264
Published by: Annals of Mathematics
Stable URL: http://www.jstor.org/stable/1969021
Accessed: 12-02-2017 22:44 UTC

ANNALS OF MATHEMATICS
Vol. 46, No. 2, April, 1945

THE DISCOVERY OF INCOMMENSURABILITY BY HIPPASUS OF METAPONTUM*

BY KURT VON FRITZ

(Received October 23, 1944)

The discovery of incommensurability is one of the most amazing and far-reaching accomplishments of early Greek mathematics. It is all the more amazing because, according to ancient tradition, the discovery was made at a time when Greek mathematical science was still in its infancy and apparently concerned with the most elementary, or, as many modern mathematicians are inclined to say, most trivial, problems, while at the same time, as recent discoveries have shown, the Egyptians and Babylonians had already elaborated very highly developed and complicated methods for the solution of mathematical problems of a higher order, and yet, as far as we can see, never even suspected the existence of the problem.

No wonder, therefore, that modern historians of mathematics have been inclined to disbelieve the ancient tradition which dates the discovery in the middle of the 5th century B.C.,[1] and that there has been a strong tendency to date the event much later, even as late as the first quarter of the 4th century.[2] But the question can hardly be decided on the basis of general considerations. It is the purpose of this paper to prove: 1) that the early Greek tradition which places the second stage of the development of the theory of incommensurability in the last quarter of the 5th century, and therefore implies that the first discovery itself was made still earlier is of such a nature that its authenticity can hardly be doubted, 2) that this tradition is strongly supported by indirect evidence, 3) that the discovery can have been made on the 'elementary' level which, even

---

* This article owes much to discussions of the early history of Greek mathematics which were carried on more than ten years ago between the author and Professor S. Bochner, now of Princeton University. This does not mean, of course, that Dr. Bochner has any part in whatever deficiencies the present article may have.

[1] This tradition will be discussed below, pp. 244 ff.

[2] The first to make an attempt to show that the discovery of incommensurability was 'late,' and certainly later than ancient tradition indicates, was Erich Frank in his book on *Platon und die sogenannten Pythagoreer* (Halle, Max Niemeyer, 1923). He does not commit himself to a definite date, but contends that the discovery cannot have been made before the last years of the 5th century (p. 228 ff.). O. Neugebauer, the most outstanding living authority on the earliest history of mathematics, goes even farther. In a letter to the author of the present paper he expressed the opinion that the discovery could not have been made before Archytas of Tarentum. Since Archytas was head of the government of Tarentum in 362 B.C., this seems to indicate that in his opinion the discovery was not made before the early 4th century at the earliest. It was also he who based his opinion on the 'trivial' character of 5th century Greek mathematics. In the present paper an attempt will be made to show that Greek mathematics in that period was in fact very elementary in many respects when compared with contemporary or earlier Babylonian and Egyptian mathematics, but by no means 'trivial.'

according to E. Frank and O. Neugebauer,[3] Greek mathematics had reached in the middle of the 5th century, 4) that the character of scientific investigation as developed in the early part of the 5th century makes it not only possible but very probable that the discovery was made at the time in which the late ancient tradition places it, and 5) that this late tradition itself contains some hints as to the way in which the discovery, in all likelihood, actually was made.

The earliest precise and definite tradition concerning a phase in the development of the theory of incommensurability is found in Plato's dialogue *Theaetetus*, p. 147 B. This dialogue was written in the year 368/67 B.C., shortly after the death of the mathematician Theaetetus after a battle in which he had been fatally wounded.[4] The fictive date of the dialogue is the year 399 B.C., that is, the year of the death of Socrates. In the first part of the dialogue the old mathematician Theodorus of Cyrene is represented as demonstrating to a group of young men, among them young Theaetetus, who is represented as a youngster of about seventeen, the irrationality of the square roots of 3, 5, 6, etc. up to 17. Though the dialogue itself is, of course, fictive, it seems hardly possible to assume that Plato, in a dialogue dedicated to the memory of a friend who has just died prematurely and who had had a very important part in the development of the theory of incommensurability and irrationality[5] would have attributed to someone else what was really his friend's own accomplishment. The inevitable conclusion, therefore, is that what Theodorus demonstrates in the introduction to the dialogue was actually known when Theaetetus was a boy of seventeen.[6]

Theodorus of Cyrene is represented as an old man in Plato's dialogue. According to an extract from Eudemus' history of mathematics[7] he was a contemporary of Hippocrates of Chios and belonged to the generation following that of Anaxagoras and preceding that of Plato. Since Anaxagoras was born in ca 500, and Plato in 428, this implies that Theodorus was born about 470 or 460, which agrees with Plato's statement that he was an old man in 399. Plato

---

[3] See the preceding note.

[4] This was proved by Eva Sachs in her dissertation *De Theaeteto mathematico* (Berlin, 1914). Her results in this respect seem absolutely certain and have been universally accepted.

[5] For details see my article *Theaitetos* in Pauly-Wissowa, *Realencyclopädie*, vol. V A, p. 1351–72.

[6] E. Frank (*op. cit.*, pp. 59, 228, and passim) and others have quoted a passage in Plato's *Laws* (p. 819c ff.) as a proof of their assumption that the discovery of incommensurability cannot have been made before the end of the fifth or the beginning of the fourth century. In this passage 'the old Athenian,' who is usually identified with Plato, says that he became acquainted with the discovery of incommensurability only late in his life and that it is a shame that 'all the Greeks' are still ignorant of the fact. It is quite clear that the latter statement is a rhetorical exaggeration since 'all the Greeks,' if taken literally, would include the Athenian himself, who by now obviously does know. The passage then proves nothing but that even striking mathematical discoveries in the fifth century did not become known to the general educated public. But this is also true of the fourth and third centuries.

[7] In Proclus' commentary to Euclid's Elements, p. 66 Friedlein.

does not say that what Theodorus demonstrated to Theaetetus and the other youngsters in 399 was at that time an entirely new discovery, though the fact that he gave a proof for each one of the different cases separately shows that the theory had not yet reached a more advanced stage.[8] But even if we assume that Theodorus' demonstrations had been worked out for the first time not so very long before, Plato's dialogue would still indicate that the irrationality of the square root of 2, or the incommensurability of the side and diameter of a square had been discovered by someone else. For it is difficult to see why he should have made Theodorus start with the square root of 3, unless he wished to give an historical hint that this was the point where Theodorus' own contribution to mathematical theory began. This in itself then would be quite sufficient to show that the discovery of incommensurability must have been made in the earlier part of the last quarter of the 5th century at the very latest, and since mathematical knowledge at that time traveled very slowly, may very well have been made earlier.[9]

What can be inferred from Plato's dialogue *Theaetetus* receives strong confirmation from indirect evidence which has been presented by H. Hasse and H. Scholz.[10] It is perhaps not necessary to accept their interpretation of the doctrines of Zenon of Elea in every detail. But there can hardly be any doubt that they have proved conclusively that there must have been a connection between some of Zeno's famous arguments against motion, and the discovery of incommensurability.[11] Since Zenon was born not later than 490 B.C., acceptance of the results of the treatise quoted would lead to the conclusion that the discovery of incommensurability must have been made not later than the middle of the 5th century, which is also the date indicated by ancient tradition.

In contrast to the tradition concerning the second phase of the development of the theory of incommensurability the tradition concerning the first discovery itself has been preserved only in the works of very late authors, and is frequently connected with stories of obviously legendary character.[12] But the tradition is

---

[8] Concerning the probable steps from the first discovery to the theory of Theodorus, see infra pp. 254 ff.

[9] See note 6.

[10] H. Hasse and H. Scholz, *Die Grundlagenkrisis der griechischen Mathematik*, Charlottenburg, Kurt Metzner, 1928, pp. 10 ff.

[11] In contrast to this, E. Frank (*op. cit.*, pp. 219 ff.) has contended that the mathematical philosophy of the Pythagoreans which *preceded* the discovery of incommensurability presupposes the atomistic theory of Democritus and a fully developed theory of 'the subjectivity of sensual qualities.' The analysis of the early form of Pythagorean philosophy attempted below will, I hope, show that it has nothing whatever to do with Democritus' atomism, and is certainly no more dependent on a fully developed theory of the subjectivity of sensual qualities than the philosophy of Parmenides, who was born at least 60 years earlier than Democritus.

[12] For instance, the story told by Iamblichus, that he was drowned in the sea, and that this was a divine punishment for his having made public the secret mathematical doctrines of the Pythagoreans.

unanimous[13] in attributing the discovery to a Pythagorean philosopher by the name of Hippasus of Metapontum.

Ancient tradition concerning the life and chronology of Hippasus is scanty. Iamblichus in his treatise *de communi mathematica scientia*[14] says that early Greek mathematical science made great progress through the work of Hippocrates of Chios and Theodorus of Cyrene, who followed upon Hippasus of Metapontum. Since Hippocrates and Theodorus are also mentioned together in the extract from the history of mathematics of Eudemus of Rhodes,[15] it seems likely that Iamblichus' note also goes back to the very reliable work of this disciple of Aristotle. According to this work Hippasus belonged to the generation preceding that of Theodorus (according to ancient usage this means an average difference of age of about 30–40 years), who in his turn was a contemporary of Hippocrates of Chios.

According to Iamblichus' Life of *Pythagoras*,[16] Hippasus had an important part in the political disturbances in which the Pythagorean order became involved in the second quarter of the 5th century, and which ended in the revolt of ca 445, which put an end to Pythagorean domination in southern Italy.[17] This agrees perfectly with the tradition which places him in the generation before Theodorus, who, as shown above, was born between 470 and 460. This confirmation is all the more valuable because the tradition of the political history of the Pythagoreans which was first collected by Aristoxenus of Tarentum and Timaeus of Tauromenium is, on the whole, quite independent from the ancient tradition of early Greek mathematics, which was first collected by Eudemus of Rhodes.

The mathematical achievements—apart from the discovery of incommensurability—ascribed to Hippasus by ancient tradition, are the following:

1. An anonymous scholion on Plato's *Phaedo*,[18] quoting a work on music by Aristotle's disciple Aristoxenus, says that Hippasus performed an experiment with metal discs. He had four metal discs of equal diameter made in such a way that the second disc was $1\frac{1}{3}$ times as thick, the third $1\frac{1}{2}$ times as thick, and the fourth twice as thick as the first one. He then showed that by striking any two of them the same harmony of sounds would be produced as by two strings whose lengths were in the same proportion as the thicknesses of the discs. Theon

---

[13] The one seeming deviation from the unanimous tradition in Proclus, *op. cit.* (see note 7), p. 67, is obviously due to a corrupt reading (*ἀλόγων* for *ἀναλόγων* or *ἀναλογιῶν*) in some manuscripts.

[14] Iamblichus, *De communi mathematica scientia*, 25, p. 77 Festa.

[15] See note 7.

[16] Iamblichus, *De Vita Pythagorae*, 257, p. 138 f. Deubner.

[17] For the date see K. von Fritz, *Pythagorean Politics in Southern Italy* (Columbia University Press, 1940), pp. 77 ff.

[18] Schol. in Plat. Phaed. 108d; see *Scholia Platonica*, ed. W. Chase Greene (Philol. Monographs publ. by Am. Philol. Ass., vol. VIII, 1938), p. 15. All the passages quoted in notes 18–24 are also collected, though sometimes in a slightly abbreviated form, in H. Diels, *Vorsokratiker*, Vol. 1.

of Smyrna[19] attributes to him a similar experiment with four tumblers, the first of which was left empty, while the others were filled $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$ with water.

2. Boethius[20] attributes to him a theory of the musical scale showing how the different musical harmonies can mathematically be derived from one another.

3. Iamblichus[21] says that Hippasus concerned himself with the theory of proportions and 'means' and was the first to change to 'harmonic mean' the name of what previously had been called the contrary, or, as some translate, the subcontrary, mean, the formula of which is $\frac{a}{c} = \frac{a-b}{b-c}$. But Nicomachus attributes this change in terminology to Philolaus.

4. According to Iamblichus,[22] Hippasus was also the first to draw or construct[23] the 'sphere consisting of 12 regular pentagons', or, as he says in another passage,[24] to inscribe the regular dodecahedron in a sphere and to make this construction public, which was considered a criminal divulgation of Pythagorean secret knowledge.

Of these four statements the first and fourth are of special importance and must be carefully analyzed, while the second and the third are of a certain importance for our problem mainly in connection with the first one.

In regard to Hippasus' experiments it seems relevant to point out that in the period in which Hippasus lived other Greek philosophers also conducted scientific experiments, while after that time, with one possible exception,[25] we do not again hear of scientific experiments until the third century. In fact, the philosopher to whom most of these experiments are attributed, Empedocles (ca 490 to ca 430 B.C.), was a native of Sicily, lived for some time in southern Italy, and though not a Pythagorean himself, was undoubtedly influenced by Pythagorean thought.

The experiments attributed to Empedocles are the following: 1) an experiment to show that drinkable water could be extracted from the sea, in order to show that fish did not 'feed on' salt water, but on sweet water which could be extracted from it;[26] 2) an experiment with small open vessels filled with water and swung around on a cord, in order to prove the existence of what we would call a centrifugal force, which in his opinion prevented the celestial bodies from falling to the earth;[27] 3) an experiment with pulverized ore of various kinds and colors,

---

[19] Theo Smyrnaeus, *Expos. Rerum Mathem.*, p. 59 Hiller.

[20] Boethius, *De Institutione musica*, 11, 10.

[21] Iamblichus, *In Nicomachi arithmet. introd.*, p. 109 Pistelli.

[22] Iamblichus, *De communi mathem. scientia* 25 (p. 77 Festa) and *Vita Pythag.* 18, 88 (p. 52 Deubner).

[23] The Greek term *γράψασθαι* has both meanings.

[24] *Vita Pyth.* 34,247 (p. 132 Deubner). The name of Hippasus is not mentioned in this passage, but since the same story is connected with the divulgation of the discovery as in the first passage, there can be no doubt that the reference is to Hippasus.

[25] See infra.

[26] Empedocles, fragm. A 66 in H. Diels, *Die Fragmente der Vorsokratiker*, vol. 1.

[27] Ibid., A 67.

in order to show that the different elements when mixed in this way become inseparable, and their original qualities indistinguishable in the mixture;[28] 4) an experiment with a *clepsydra* or water-clock, in order to prove that seemingly completely empty vessels are actually filled with air.[29] This experiment and a similar one with leather bags is also attributed to Anaxagoras[30] (born in ca 500 B.C.).

The one possible exception to the statement that the known scientific experiments of the Greeks belong to the fifth and third (and later) centuries, but not to the fourth, is found in a passage from a work of Archytas, quoted literally by Nicomachus and Porphyrius.[31] In this fragment Archytas propounds the theory that sound is produced by a concussion of the air, that the pitch of the sound depends on and is proportional to the velocity of the motion producing it, and that if the velocities producing two sounds are in certain simple numerical ratios, well known musical harmonies result. The arguments by which these theories are supported are based on observations which *can* be made in everyday life, and without experimentation; but the way in which the observations are introduced strongly suggests that, though originally they may have been made incidentally, they were at least checked by being repeated in an experimental fashion. Archytas, however, does not claim to be the author of these theories and to have made personally the observations or experiments from which they are derived, but attributes them to mathematicians whose names he does not give. At the same time it is obvious that these theories and observations represent an advanced stage of scientific development as compared with the experiments of Hippasus and their results. For in the Archytas fragment Hippasus' demonstration of a way in which the same musical harmonies can be produced by any conceivable kind of sound-producing instrument is integrated with a general physical theory of sound. Since, on the other hand, both Hippasus and Archytas were Pythagoreans living in southern Italy, since Archytas, as shown above,[32] belonged to the second generation after Hippasus, and since, nevertheless, Hippasus and Archytas are sometimes mentioned together in ancient tradition[33] as having contributed to the development of a physical theory of sound, there really seems to be no reason to doubt that there actually existed a scientific tradition in one branch of the Pythagorean school through which a theory of sound was gradually developed. Since, finally, the authenticity of the fragment from Archytas' *Har-*

---

[28] Ibid., A 34.

[29] Ibid., B 100. Here the description of the experiment is given in its original wording. Empedocles in fact does not describe it as an experiment made by himself, but as an illustrative analogy derived from the observation of a young girl playing with a water-clock. But this belongs to the poetical style, since Empedocles expounded all his philosophical and scientific theories in verse. The minute description of the process leaves no doubt whatever that Empedocles must have made the experiment himself.

[30] Anaxagoras, fragm A 68/69 in H. Diels, *op. cit.*

[31] Archytas, fragm. B 1 (Diels, *op. cit.*).

[32] See supra p. 245 and note 2.

[33] For instance, Iamblichus, in *Nicom. arithm. intr.*, p. 109 Pistelli.

monikos can hardly be doubted, and as far as I can see never has been doubted, and since he clearly implies that the theory of sound had reached a rather advanced stage before he himself began to contribute to it, it is difficult to see how some scholars[34] could claim that ancient tradition projected into a much earlier time the accomplishments of a later period, when it attributed to Hippasus, a man belonging to the second generation before Archytas, the first beginnings of a theory which had reached a much more advanced stage before Archytas wrote his work.

Everything then seems to confirm the assumption that the experiments attributed to Hippasus by ancient tradition actually can have been made, and most probably were made, in Southern Italy in the middle of the fifth century, that is, when Hippasus is supposed to have lived in that region. To that extent, at least, the late tradition, which according to E. Frank and others, is of no value whatever, seems to be vindicated.

But what can Hippasus' experiments with discs and tumblers possibly have to do with the discovery of incommensurability? In order to show the interconnection, which is, of course, very indirect, it will be necessary to make a further analysis of the purpose and meaning of these experiments.

All the experiments ascribed to philosophers of the fifth century, as their description clearly shows, were obviously undertaken not so much in order to find out something new, but rather in order to support and verify an already existing theory, for instance, that the fish do not consume salt water as such, but extract sweet water from it, that the celestial bodies do not remain in the sky because they are lighter than air, etc. The same is true of the experiments attributed to Hippasus. That certain musical harmonies would be produced if the lengths of two strings of the same kind were in certain ratios had always been known. It had also been known in regard to flutes. From this double knowledge, then, the general assumption was derived that it would be so in all cases. What Hippasus did was, in a way, nothing but a verification of this assumption by means of various sound-producing bodies which were not ordinarily used as musical instruments. But two things are significant. Strings have, so to speak, only one dimension. In regard to flutes, too, especially if the different tones are produced on the same flute, one will not always think of the other two dimensions. When Hippasus used tumblers and discs, however, he had to point out that the discs, for instance, must be equal in two dimensions and differ only in the third if the musical harmonies are to be produced, but that it did not matter whether the third dimension was what usually was called length or thickness. In this way, then, the result can be most clearly formulated, namely, that the musical harmonies are completely independent of the material of which the sound-producing body consists, and of the special quality or color of the tones produced, and that the production of these harmonics depends exclusively on simple one-dimensional numerical ratios. We hear then, further,[35]

---

[34] See E. Frank, *op. cit.*, p. 69 and passim.

[35] See supra, p. 246, note 20.

that Hippasus was not content with having proved this point but also investigated the mathematical relations between the ratios producing the most outstanding harmonies and tried to derive them mathematically from one another.

As long as Hippasus remained within the limits of the theory of music, all this, of course, could not lead to the discovery of incommensurability. But there are strong indications that he and his associates did not confine themselves to this special field.

Aristotle very frequently mentions the Pythagoreans or so-called Pythagoreans, and attributes to them the doctrine that 'all things are number.'[36] According to E. Frank these so-called Pythagoreans are not Pythagoreans at all,[37] but contemporaries of Plato who were deeply influenced by his philosophy.[38] If this were so it would be difficult to see why Aristotle, who should have known, never says a word about it, and always seems to imply that Plato's theory of numbers is later. It would also be possible to show that the comparatively very primitive Pythagorean theory cannot possibly be later than Plato's very complicated one. But this would require an analysis of considerable length, which fortunately is not necessary for the present purpose, since there is more direct evidence to show that there must have been Pythagoreans in the fifth century who had a doctrine similar to that ascribed to them by Aristotle.

Archytas in the long fragment quoted above[39] says that the same men who elaborated a theory of sound had also attained 'clear insight' into problems of astronomy, geometry, and arithmetic. Again, of course, he refers to what others had done before he wrote his work. Unfortunately, the passages in which he described the achievements of his predecessors in astronomy and geometry have not come down to us. But since he speaks of the clear insight which they had attained, it is not likely that it was only in music that they had arrived at a stage so advanced that it must have required a considerable time to attain it. Moreover, Archytas says that the sciences mentioned are intimately related to one another because all of them 'turn back' to 'the first (or fundamental) form of everything that is'. This seems a very advanced form of the doctrine which

---

[36] The doctrine is expressed and explained in a great many different ways by Aristotle; for instance, that 'the elements of numbers are the elements of all things' (Metaph. 986a, 1 ff.), or that 'all things are composed of numbers' (ibid. 1080b, 16 f.), or that 'the things themselves are numbers' (ibid., 987b, 29 f.), or that 'number is the essence of everything' (ibid., 987a, 19). But the last expression uses specific Aristotelian terminology and is obviously an attempt to explain what appeared too odd in its original wording.

[37] *Op. cit.*, p. 68 ff.

[38] E. Frank lays great stress on the fact that Aristotle speaks often, though not in the majority of cases, of the 'so-called' Pythagoreans, and infers from this that he meant that they were not really Pythagoreans. In fact, there was an excellent reason for the use of the word 'so-called,' namely, that in Aristotle's time 'Pythagoreans' was the only name designating the adherents of a philosophical school or sect that was derived from the name of the founder; that is, it was an unusual expression. Confirmation of this can also be found in the fact that the only analogy to the name 'Pythagoreans' found in pre-Aristotelian literature (Herakleiteans in Plato's Theaet. 179e) is obviously used in fun.

[39] See supra p. 247 and note 31.

Aristotle attributes to the 'so-called Pythagoreans'. Again, everything seems to indicate that the close connection between arithmetic, geometry, astronomy, and musical theory, as well as the somewhat crude theory that 'all things are numbers' must have been considerably older than Archytas, that is, at least as early as the middle of the fifth century.

In order to understand the origin and meaning of this latter doctrine, an analysis of the Greek terminology of the theory of proportions will be helpful. The Greek expression for proportion means literally 'the same ratio'. For our term 'ratio' the Greeks have two expressions: *diastema*, which means literally 'interval', and *logos*, which means literally 'word'. The first term clearly shows the connection of the early theory of proportions with musical theory.[40] But the second term is even more significant. The Greeks had two terms for 'word': *epos* and *logos*.[41] *Epos* means the spoken word, or the word which appeals to the imagination and evokes a picture of things or events. This is the reason why it is also specifically applied to epic poetry. *Logos* designates the word or combination of words in as much as they convey a meaning or insight into something.[42] It is this connotation of the term *logos* which made it possible for it in later times to acquire the meaning of an intrinsic law or the law governing the whole world.

If *logos*, then, is the term used for a mathematical ratio, this points to the idea that the ratio gives an insight into a thing or expresses its intrinsic nature. In the case of musical harmonies the harmony itself would be perceived by the ear, but it was the mathematical ratio which, in the mind of the Pythagoreans, seemed to reveal the nature of the harmony, because through it the harmony could be both defined and reproduced in different media.

It is easy to see how this general idea could be extended to astronomy, especially to the regular motions of the celestial bodies and the interrelations between their various cycles.[43] But it is the extension of the theory to geometry which is of special importance for our problem.

The mathematical theorem which is in tradition most closely connected with Pythagoras and the Pythagoreans, is the theorem that in a right-angled triangle the sum of the squares on the sides including the right angle is equal to the square

---

[40] This is also the case with the word *horos* designating the terms of a ratio or a proportion. See K. von Fritz, *Philosophie und sprachlicher Ausdruck bei Demokrit, Platon und Aristoteles* (New York, Stechert, 1938), p. 69.

[41] As to the question of how early the term *logos* was used in the sense of ratio, see infra p. 261 f.

[42] This is also characteristic of the corresponding verb *legein*. In consequence, the Greeks can form the following sentence: N. N. says (there follows a literal quotation of his words) saying (there follows an interpretation of their meaning). It is clear that 'saying' in this sentence really means 'meaning.' The verb *eipein*, which corresponds to *epos* cannot be used in the latter sense. It is also significant that those stories which Herodotus, for instance, calls *logoi* are always stories with a moral, that is, with a meaning.

[43] For details see my article on Oinopides of Chios in Pauly-Wissowa *Realencyclopaedie*, vol. 17, p. 2260-67.

on the side subtending the right angle. Nobody who knows anything about the early history of Greek mathematics has ever doubted that the proof of this theorem given by Euclid in the first book of his Elements cannot have been found by Pythagoras or his early followers. This is also what the best ancient tradition says, since Proclus attributes this proof to Euclid himself.[44] Though at the time when, in the last quarter of the fifth century, Hippocrates of Chios elaborated his famous theory of the *lunulae*, the 'Pythagorean theorem' must have been considered valid for right-angled triangles whose sides are commensurable with one another and for triangles whose sides are incommensurable, and furthermore must have been extended to cover all similar figures erected on the sides of a right-angled triangle, it is not possible for us to find out exactly how the early Greek mathematicians proved or tried to prove the theorem in this general form, since there exists no tradition about it.[45] Fortunately, it is not necessary for our purpose to have this knowledge.

Again, the theory must have started from an observation which had been generally known long before the beginning of Greek philosophy, namely, that if one puts together three pieces of wood of the respective lengths of 3, 4, and 5, a right-angled triangle will result. In fact, this is an old form of a carpenter's square. Since the size of carpenter's squares was not standardized, it must also have been a matter of common knowledge that the absolute length of the sides of the triangle was irrelevant, and that all triangles whose sides were in that proportion were not only right-angled, but also 'similar' in shape. Finally, it seems to have been known of old that the sum of the squares of 3 and 4 was equal to the square of 5.

Even if we had no tradition about it we would have to conclude that the Pythagoreans must have been impressed by these facts as soon as they had begun to suspect that the nature of a good many things might be found in or expressed by numbers, especially since there is indirect evidence to show that even before Pythagoras the philosopher Thales (ca 620 to ca 540 B.C.) and his followers had concerned themselves with what we may call the ornamental shape of geometrical figures[46] and also seem to have connected this ornamental appearance especially with the angles. The fact, at least, that according to Proclus[47] they used the term 'similar angles' for what later was called 'equal' angles can hardly be explained otherwise.[48]

On the basis of this earlier development the Pythagoreans can hardly have

---

[44] Proclus, *In primum Euclid. elem. librum Comment.*, p. 426 Friedlein.

[45] For the various possibilities see the lucid exposition of Th. Heath in his commentated translation of Euclid's *Elements* (Cambridge 1926), vol. 1, pp. 352 ff.

[46] For the evidence see Th. Heath, *A History of Greek Mathematics* (Oxford, 1921), vol. 1, pp. 130 ff.

[47] *Op. cit.*, p. 250 Friedlein.

[48] It is perhaps pertinent to observe that the historian Thucydides (I, 77) also uses the term 'similar' where he means equality of form (or in this case: procedure). For he uses the expression 'similar laws' where, as the context shows, he does not mean similar laws but what otherwise was called *isonomia* or equality before the law.

failed to notice that any two triangles will be similar in shape if their sides are in proportion, though in actual fact in the earliest period this knowledge can have been an exact knowledge only in regard to triangles whose sides are commensurable with one another. Though this assumption is not supported by any direct tradition—probably because it was too obvious to be especially mentioned—it follows not only from the general situation, but especially from the close analogy of the Pythagoreans' theory of music and their earliest theory of geometrical figures, which is attested everywhere. For just as they declared that the musical harmonies which are perceived by the ear 'are' really the numbers by which the proportionate lengths of the strings, etc., producing them are measured, so the geometrical figures, whose shape is perceived by the eye but cannot otherwise be either exactly determined or expressed in language, 'are' really the numbers or sets of numbers constituting the ratios of the lengths of their sides by which their shape is determined and can therefore be expressed.[49]

According to ancient tradition, the theory, before the discovery of incommensurability, was further extended in two directions. Proclus[50] credits Pythagoras with a formula which makes it possible to form any number of different rational right-angled triangles by finding pairs of numbers the sum of the squares of which is equal to a square number.[51] It is irrelevant for our purpose whether this formula is rightly attributed to Pythagoras personally, but one can safely assume that it belongs to the very oldest period of Pythagorean mathematics. For Proclus usually relies on the very excellent history of mathematics of Aristotle's disciple Eudemus of Rhodes; and in this case what he says seems all the more worthy of credit in that he does not claim too much and rather implies a criticism of the common tradition that Pythagoras 'proved' the 'Pythagorean theorem' in its general geometrical form.

Nevertheless, the formula marks a great advance. One has to interpret it in terms of Pythagorean philosophy in order to understand its importance in regard to our problem. In the theory discussed before, the shape of figures which are similar in the mathematical sense of the word is directly related to a definite set of integers. Two triangles, with the sides 3, 4, 5 and 8, 15, 17 respectively are not, on the other hand, *similar* in the sense of the (modern or Euclidean) mathematical term. But they are still 'similar' in regard to the ornamental element of one right angle; and this 'similarity' is not related to or expressed in one definite set of integers, but is related to the fact that the two

---

[49] The Pythagoreans were, of course, aware that triangles are the only rectilineal figures whose shape is definitely determined by the proportionate length of their sides. That they realized the importance of this fact for their theory seems proved by Theon's statement (*op. cit.*, pp. 40 ff.) that they divided all other rectilineal figures into triangles.

[50] *Op. cit.* (see note 44), p. 428 Friedlein.

[51] The formula, though expressed in a somewhat more complicated way amounts to the statement that if m be any odd number,

$$m^2 + \left(\frac{m^2 - 1}{2}\right)^2 = \left(\frac{m^2 + 1}{2}\right)^2.$$

sets of integers related to the two triangles enter into the same mathematical formula. What is important for our problem in this extension of the theory is merely that it shows how the Pythagoreans were not content with a simple theory but, with an extraordinarily inquisitive spirit, adapted this theory to ever more complicated problems.

The second extension of the Pythagorean theory which is important as a preparation for the discovery of incommensurability is the theory of polygonal numbers. This theory, the beginnings of which ancient tradition, starting with Aristotle,[52] attributes also to the early Pythagoreans, was many centuries later developed by Diophantus to what is now called indeterminate analysis. But for a long time it remained rather sterile from a purely mathematical point of view. This is probably the reason why Euclid disregarded it in the arithmetical section of his elements and why other high ranking mathematicians from the fourth century onwards have done likewise.

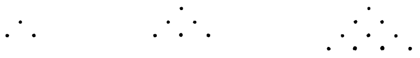
Just like the other geometrical theories of the Pythagoreans discussed so far, this theory is concerned with interrelations between numbers and geometrical figures. But in this case the figures are not drawn and formed by straight lines of certain proportionate measures, but are built up from dots. The theory then is concerned with the question from what numbers of dots arranged in a certain order the different polygons can be built.[53] It seems perfectly clear from the evidence presented so far that this theory is a natural product of the development of Pythagorean thought. It, therefore, certainly need not, as E. Frank contends,[54] be dependent on or, in its original form, even be influenced by the physical atomism of Democritus, which has an entirely different origin. Whatever chronological inferences E. Frank draws from this incidental affinity are, therefore, absolutely unwarranted.[55]

Though the 'atomism' of the theory of polygonal numbers seems most remote from the discovery of incommensurability it is here that we come nearest to our problem. All the Pythagorean doctrines discussed so far either are based on or result in a search for numbers, i.e., integers, from which geometrical figures with certain properties can be built up. In the course of these efforts the Pythagoreans can hardly have failed to wonder what numbers might be hidden in certain

---

[52] The relevant passages have been collected by Heath, *History* (see note 46), 1, 76 ff.

[53] Triangular numbers, for instance, are (1), 3, 6, 10, 15, like this:



etc.

[54] *Op. cit.* (see note 2), p. 52 ff.

[55] The passage in Aristotle, *De Anima*, 490a, 10 ff., where Aristotle quite correctly says that if one replaces Democritus' material atoms by immaterial dots the result is very similar to the quantitative theory of the Pythagoreans, need certainly not have chronological implications. But even if it had such implications, this would not prove anything, since Aristotle in this passage does not refer to the earliest form of the Pythagorean doctrine.

well-known figures which had not been built up in this way, for instance, the isosceles right-angled triangle, which was of special importance to the Pythagoreans because it was one-half of the square, the latter figure having become a mystical symbol in the Pythagorean community. In the case of the isosceles right-angled triangle, however, it is not possible to express the ratio between its sides in integers. It is perhaps not too far-fetched a speculation if one assumes that the early development of the theory of polygonal numbers was partly due to an attempt to overcome this difficulty by building up the polygons from dots rather than from straight lines. In fact, this seems all the more likely because here again the division of polygons and polygonal numbers into triangles and triangular numbers is one of the main points of the theory. Theon, for instance, points out[56] that an oblong number can be divided into two equal triangular numbers while a square number is made up of two unequal triangular numbers whose sides differ by one unit, namely,

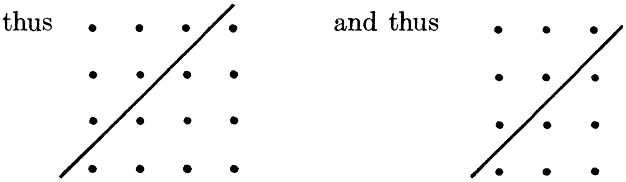


FIG. 1

But however this may be, men of the inquisitive spirit which characterized Hippasus and some of his Pythagorean contemporaries[57] can hardly have been satisfied with these arithmetical theorems as a substitute for the solution of the real problem, namely, the problem of the ratio between the sides of an isosceles right-angled triangle. This is again confirmed by ancient tradition; for what Plato says about Theodorus' demonstration of the irrationality of the square roots of 3, 5, 6, 7, etc. presupposes, as shown above,[58] that the irrationality of the square root of 2 had already been proved.

Fortunately, the original demonstration of the irrationality of the square root of 2 has been preserved in an appendix to the tenth book of Euclid's elements;[59] and that this demonstration is actually, at least in its general outline, the original one is attested by Aristotle. One glance at this demonstration[60]

---

[56] *Op. cit.*, p. 41 Hiller.

[57] See supra p. 245 ff. and p. 252.

[58] See supra, p. 244.

[59] Euclid, *Elementa*, X, Append. 27, p. 408 ff. (This appendix is not included in Heath's translation of Euclid's Elements).

[60] In literal translation this demonstration runs as follows: Let $ABCD$ be a square and AC its diameter. I say that AC will be incommensurable with AB in length.

For let us assume that it is commensurable. I say that it will follow that the same number is at the same time even and odd. It is clear that the square on AC is double the square on AB. Since then (according to our assumption) AC is commensurable with AB, AC will be to AB in the ratio of an integer to an integer. Let them have the ratio $DE:F$ and let DE and F be the smallest numbers which are in this proportion to one another. DE cannot then be the unit. For if DE was the unit and is to F in the same proportion as shows that it does not presuppose any geometrical knowledge beyond the Pythagorean theorem in its special application to the isosceles right-angled triangle, which, as is well-known, can be 'proved' simply by drawing the figure in such a way that the truth of the theorem in that particular case is immediately visible.[61] Apart from this the demonstration remains in the purely arithmetical field; and since the early Pythagoreans speculated a good deal about odd and even numbers[62] the demonstration itself cannot have been beyond their reach.[63]

Yet if this demonstration of the irrationality of the square root of 2 was the only way in which incommensurability can have been discovered, one might still agree that there are good reasons for Frank's and Neugebauer's hesitation to attribute the discovery to the middle of the 5th century. The demonstration requires not only a good deal of abstract thinking, but also of strict logical reasoning. Apart from this, the labored language of the demonstration as given in the appendix in Euclid shows clearly with what difficulties the early Greek mathematicians had to struggle when elaborating a proof of this kind. In fact, this conclusion is all the more cogent because the demonstration, though somewhat more archaic in form than Euclid's own demonstrations, uses a form of presenting the argument in short concise sentences which has no parallel in Greek literature of the fifth century.[64] If, then, the proof as such, as the combined passages in Plato and Aristotle seem to indicate,[65] belongs to the fifth century,

---

AC to AB, AC being greater than AB, DE, the unit, will be greater than the integer F, which is impossible. Hence DE is not the unit, but an integer (greater than the unit). Now since $AC{:}AB = DE{:}F$, it follows that also $AC^2{:}AB^2 = DE^2{:}F^2$. But $AC^2 = 2AB^2$ and hence $DE^2 = 2F^2$. Hence DE^2 is an even number and therefore DE must also be an even number. For if it was an odd number its square would also be an odd number. For if any number of odd numbers are added to one another so that the number of numbers added is an odd number the result is also an odd number. Hence DE will be an even number. Let then DE be divided into two equal numbers at the point G. Since DE and F are the smallest numbers which are in the same proportion they will be prime to one another. Therefore, since DE is an even number, F will be an odd number. For if it was an even number the number 2 would measure both DE and F, though they are prime to one another, which is impossible. Hence F is not even, but odd. Now since $ED = 2EG$ it follows that $ED^2 = 4EG^2$. But $ED^2 = 2F^2$, and hence $F^2 = 2EG^2$. Therefore F^2 must be an even number, and in consequence F also an even number. But it has also been demonstrated that F must be an odd number, which is impossible. It follows, therefore, that AC cannot be commensurable with AB, which was to be demonstrated.

[61] For examples, see Heath, *The Thirteen Books of Euclid's Elements*, vol. 1, p. 352.

[62] See, for instance, Aristotle, *Physics*, 203a, 5 ff.; *Metaph.*, 986a, 22 ff.

[63] Concerning the arithmetical premises of this demonstration and the probable deficiencies of its original form, see my article on Theodorus of Cyrene in Pauly-Wissowa, *RE*, vol. VA, p. 1817 and 1820 ff.

[64] In order to illustrate this, one may compare the literal fragments of Zenon of Elea which show a very high degree of abstract thinking and also of close logical reasoning, but at the same time are written in a labored language with long and cumbrous sentences, while Aristotle (in the fourth century) and later writers who give an account of Zenon's theory, reproduce the same arguments in a sequence of very short sentences very similar to those found in the appendix to Euclid.

[65] Plato, *Theaetetus* p. 147B ff. and Aristotle, *Analytica Priora*, 41a, 26–31 and 50a, 37. See also supra p. 244 and p. 251.

it seems safe to assume that in its original form it was still more laborious. Most significant, however, is the fact that the whole proof, as presented, uses the terms *commensurable* and *incommensurable*, just as Theodorus did in Plato's *Theaetetus*, as something already known. This seems to presuppose that incommensurability was already known when the demonstration was elaborated.

Since the form of the proof as it appears in the appendix to Euclid may not be the original one, the form of the proof in Euclid's appendix may not be sufficient to show with certainty that when the irrationality of the square root of 2 was demonstrated, the discovery of incommensurability as such had already been made, probably in a different mathematical object. But if one considers the further evidence presented above, the suspicion that such was the case becomes very strong. For it is difficult to believe that the early Greek mathematicians should have discovered the incommensurability of the diameter of a square with its side by a process of reasoning which was obviously so laborious for them if they had no previous suspicion that any such thing as incommensurability existed at all. If, on the other hand, they had already discovered the fact in a simpler way, it is perfectly in keeping with what we know of their methods to assume that they at once made every effort to find out whether there were other cases of incommensurability. The isosceles right-angled triangle in that case was the natural first object of their further investigations.

It is at this point that the tradition concerning Hippasus' interest in the dodecahedron, or 'the sphere out of 12 regular pentagons' has to be considered. There can be no doubt that Hippasus was not the author of the mathematical construction of the dodecahedron, as Iamblichus claims in one place.[66] Quite apart from other considerations, this is proved by the fact that the better tradition implies that this was an achievement of Theaetetus,[67] who belonged to the second generation after Hippasus. And in another passage, Iamblichus[68] himself claims merely that Hippasus 'drew' the regular dodecahedron, which is probably the original tradition.

That Hippasus was interested in the dodecahedron and in the dodecahedron as a 'sphere made of 12 regular pentagons' is very likely. For regular dodecahedra occurred in Italy as products of nature in the form of crystals of pyrite.[69] With the Pythagoreans' interest in geometrical forms these crystals must certainly have attracted their attention and evoked a desire to analyze their form mathematically. In addition, we know that the Pythagoreans used the pentagram, i.e., a regular pentagon with its sides prolonged to the point of intersection,[70] as a token of recognition. It is absolutely in the character of Hippasus as we now know him that he should have tried to find out about the

---

[66] See note 24.

[67] For details see the article quoted in note 5, pp. 1364 ff.

[68] See notes 22 and 23.

[69] See F. Lindemann in *Sitz.-Berichte Akad. München, math.-phys. Klasse*, vol. 26, pp. 725 ff. Lindemann gives also evidence to show that dodecahedra were used as dice in Italy at a very early time, and that the regular dodecahedron seems to have had some religious importance in Etruria. Especially the latter fact, if known to the Pythagoreans, would naturally have increased their interest in the figure.

[70] See Lucian, *De lapsu in salutando*, 5, and schol. Aristoph. *Nubes*, 609.

numbers and ratios incorporated in the pentagram and regular pentagon. Could it then really be a mere coincidence that the same Hippasus is credited with the discovery of incommensurability and with an interest in the 'sphere consisting of 12 regular pentagons,' and that the regular pentagon is exactly the one geometrical figure in which incommensurability can be most easily proved?

How would the Pythagoreans have gone about it if they wanted to know the ratio between the lengths of two straight lines? Again, the method was an old one, known by craftsmen as a rule of thumb many centuries before the beginning of Greek philosophy and science, namely, the method of mutual subtraction,[71] by which one finds the greatest common measure. It is, of course, impossible to discover incommensurability by applying this method in the way in which craftsmen do it: with measuring sticks or measuring ropes. But if one looks at the pentagram or at a regular pentagon with all its diameters filled in—and we have seen that the Pythagoreans were interested in diameters—the fact that the process of mutual subtraction goes on infinitely, that therefore there is no greatest common measure, and that hence the ratio between diameter and side cannot be expressed in integers however great, is apparent almost at first sight. For one sees at once that the diameters of the pentagon form a new regular pentagon in the centre, that the diameters of this smaller pentagon will again form a regular pentagon, and so on in an infinite process.

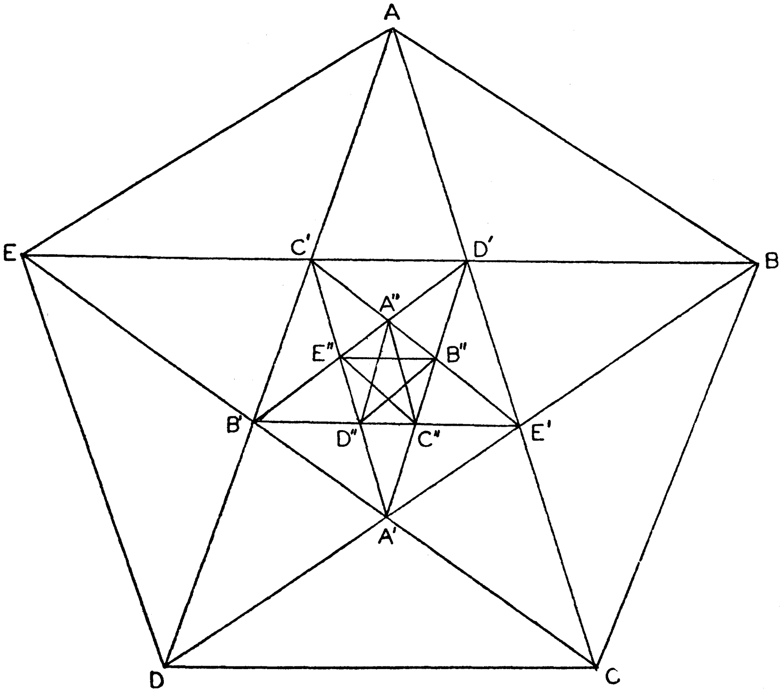


FIG. 2

---

[71] For evidence to show that the Pythagoreans used this method in mathematical theory, see infra p. 258.

It is then also very easy to see that in the pentagons produced in this way $AE = AB'$ and $B'D = B'E'$ and therefore $AD - AE = B'E'$, and likewise $AE = ED' = EA'$ and $B'E' = B'D = B'E$ and therefore $AE - B'E' = B'A'$, and so forth ad infinitum, or, in other words, that the difference between the diameter and the side of the greater pentagon is equal to the diameter of the smaller pentagon, and the difference between the side of the greater pentagon and the diameter of the smaller pentagon is equal to the side of the smaller pentagon, and again the difference between the diameter of the smaller pentagon and its side is equal to the diameter of the next smaller pentagon and so forth in infinitum. Since ever new regular pentagons are produced by the diameters it is then evident that the process of mutual subtraction will go on forever, and that therefore no greatest common measure of the diameter and the side of the regular pentagon can be found.

One may, of course, still ask how the Pythagoreans could prove that $AE = AB'$ and $B'D' = B'E'$, etc. Now Proclus, probably getting his information from Eudemus of Rhodes, states[72] that Thales was the author of the theorem that in an isosceles triangle the base angles are equal. In connection with this it is important to note that Aristotle[73] refers to an archaic proof of this proposition. He does not quote all the steps of this proof, but what he quotes shows that 'mixed angles,' i.e., angles formed by a straight line and the circumference of a circle, were used in the demonstration, and that in all likelihood the proof was based on a rather primitive method of superimposition.[74] It is clear that with this latter method the converse of the proposition could be proved without difficulty. It follows that the equality of AE with AB' and of $B'D$ with $B'E'$ could be derived from the equality of $\angle AEB'$ with $\angle AB'E$ and of $\angle B'DE'$ with $\angle B'E'D$, if these angles could be proved to be equal respectively.

As to this latter proof, the evidence is somewhat less definite. But Eudemus of Rhodes[75] attributes to the early Pythagoreans the proof that the sum of the internal angles in any triangle is equal to two right angles. From this theorem the general theorem that in any polygon the sum of the internal angles is equal to $2n - 4$ right angles can very easily be derived, if one divides the polygon into triangles,[76] and we know[77] that the Pythagoreans constantly experimented with dividing polygons into triangles. The proposition furthermore that in any polygon the sum of the external angles is equal to four right angles is a mere corollary of the preceding proposition.[78] On the basis of these propositions, finally,

---

[72] *Op. cit.*, p. 250 f. Friedlein.

[73] Aristotle, *Analyt. Pr.*, 41 b, 13 ff.

[74] For details see Heath, *Elements* (see note 45), 1, 253.

[75] Quoted by Proclus, *op. cit.*, 379 Friedlein.

[76] The proof is quoted by Proclus, *op. cit.* After the polygon has been divided into triangles, the proposition about the sum of the angles of a triangle being known, the remainder of the proof is a simple addition.

[77] See supra, p. 252, note 49.

[78] Aristotle refers to this proof as to something very well known in *Analyt. Post.* 99a, 19 ff and 85b, 38 ff.

the equality of the angles figuring in the demonstration suggested above can be very easily shown.

It follows that there is no reason whatever to disbelieve that Hippasus was able to demonstrate the incommensurability of the side with the diameter of a regular pentagon. For what is needed for the proof suggested is nothing but two fundamental geometrical propositions which concern the isosceles triangle and the sum of the angles in any triangle, and in addition the old time-honored method of finding the greatest common measure by mutual subtraction. All the rest is nothing but the simplest addition, subtraction and division. Of the two geometrical propositions, the first had undoubtedly been 'proved' in a very primitive way even before Pythagoras.[79] The second one was probably also proved in some such way, though we do not know exactly how.[80] But there can be no doubt whatever that its truth was known long before Hippasus. That the proofs of these theorems as existing in the middle of the fifth century did not come up to the Euclidean conception of a satisfactory proof is not to the point. For the question is not whether Hippasus could give a demonstration which in all its steps would have satisfied Euclid or Hilbert, but whether he was able to find a proof which at the level which mathematical theory had reached in his time was considered absolutely convincing, and as to this there can be no doubt. It is, perhaps, not unnecessary to point out specifically that the demonstration of incommensurability suggested above does not presuppose any geometrical construction in the strictly mathematical sense at all, as long as the Pythagoreans were able to draw a reasonably accurate regular pentagon in some way, and this can hardly be questioned, for a quite beautiful pentagram can be seen on a vase of Aristonophus which belongs to the seventh century B.C. This vase was found at Caere in Italy and is now in a museum in Rome. Neugebauer's argument, therefore, that the discovery of incommensurability could not have been made

---

[79] It is an interesting fact that all the theorems which ancient tradition attributes to Thales are either directly concerned with problems of symmetry and 'provable' by superimposition, or of such a kind that the first step of the proof was obviously based on a consideration of symmetry and the second step, which brings the proof to its conclusion, is a simple addition or subtraction. The much discussed Euclidean proof of the first theorem of congruence by superimposition seems, then, the last remnant of a method which once had been widely applied and with which Greek scientific geometry had started.

[80] The proof attributed to the Pythagoreans by Eudemus seems to presuppose the famous fifth postulate of Euclid. But Aristotle (*An. Pr.*, 65a, 4) indicates that there existed an old mathematical demonstration about parallels and angles which involved a vicious circle. It seems, then, quite possible that the equality of alternate angles on parallels cut by a straight line was at first considered self-evident on the basis of considerations of symmetry, that then a faulty attempt to prove the proposition was made, and that finally Euclid tried to give the whole theory a sound foundation by his famous postulate. In this case the proof of the proposition concerning the sum of the angles of a triangle attributed to the early Pythagoreans by Eudemus may really be very old. But Geminus (in Eutocius' commentary on the *Conica* of Apollonius of Perge, vol. II, 170 of Heiberg's ed. of Apoll.) mentions a still older demonstration in which the proposition was proved first for the equilateral, then for the isosceles, and finally for the scalene triangle.

by Hippasus since Oinopides, who belonged to the succeeding generation, was still concerned with the most 'trivial'[81] mathematical constructions, has no validity.

There is, then, perhaps some justification for the claim that the analysis so far has proved what was promised in the introduction to this paper, namely, that the discovery of incommensurability can have been made in the middle of the fifth century, that the development of the Pythagorean doctrine of numbers as the essence of everything naturally led to this discovery, that ancient tradition contains strong hints as to the way in which the discovery actually was made, and last but not least, that Greek mathematics in that early period may have been very elementary,[82] but certainly was not trivial. It was not trivial because the Greeks had two peculiarities which the Egyptians and Babylonians obviously lacked. They were very prone to build up sweeping general theories on very scanty evidence. Of this the Pythagorean theory that 'all things are numbers' is a striking example. Yet at the same time they were not content with having such a theory, but made unremitting efforts to verify it in all directions. It was on account of this second peculiarity that they discovered incommensurability in a very early period.

It is perhaps advisable to add a brief survey of the immediate consequences of the discovery of incommensurability for the further development of the theory of proportions. For this will confirm both the opinion concerning the general character of the early scientific investigations of the Greeks and some special suggestions which have been made in the course of the present inquiry.

The discovery of incommensurability must have made an enormous impression in Pythagorean circles because it destroyed with one stroke the belief that everything could be expressed in integers, on which the whole Pythagorean philosophy up to then had been based. This impression is clearly reflected in those legends which say that Hippasus was punished by the gods for having made public his terrible discovery.

But the consequences of the discovery were not confined to the field of philosophical speculation. *Logos* or ratio, as we have seen,[83] meant the expression of the essence of a thing by a set of integers. It had been assumed that the essence of anything could be expressed in that way. Now it had been dis-

---

[81] In my article on Oinopides (see note 43) I have tried to show that Oinopides' mathematical constructions were not 'trivial' either, if viewed in connection with the problems which he tried to solve. But the solution of the present problem is quite independent from the acceptance or rejection of this suggestion.

[82] In the present article only so much mathematical knowledge has been attributed to the early Greek mathematicians as can be ascribed to them with the greatest approximation to certainty which a historical inquiry can attain. The attempt has then been made to show that *even if* their knowledge did not go beyond this, they nevertheless can have discovered incommensurability and by the nature of their theories and methods were naturally led to this discovery. But this does not imply that their knowledge must necessarily have been as limited and elementary as has been assumed in this paper.

[83] See supra p. 249/50.

covered that there were things which had no *logos*. When *we* speak of irrationality or incommensurability we mean merely a special quality of certain magnitudes in their relation to one another, and we speak even of a special class of irrational numbers. But when the Greeks used the term *alogos*, they meant originally, as the term clearly indicates, that there was no *logos* or ratio.

Yet this fact must have been very puzzling. It had been generally assumed that two triangles which were similar, i.e., which had the same ornamental appearance, though differing in size, had the same *logos*, i.e., that they could be expressed by the same set of integers. In fact, this is clearly the original meaning of the term *ho autos logos* (the same *logos*), which we translate by 'proportion.' But two isosceles right-angled triangles had still the same ornamental appearance, and therefore should have had the same *logos*. In fact, it seemed evident that their sides did have the same quantitative relation to one another. Yet they had no *logos*.

The way in which the Greeks amazingly soon after the stunning discovery of incommensurability began to deal with this problem is a much greater proof of their genius for and their tenacity in the pursuit of scientific theory than the discovery of incommensurability itself. For very soon[84] they began not only to extend the theory of proportion to incommensurables, but also established a criterion by which in certain cases it can be determined whether two pairs of incommensurables (which in the old sense have no *logos* at all) have the same *logos*. The terminological difficulty created by this seeming contradiction in terms is reflected by the fact that for some time the term *alogos* for irrational was replaced by the term *arrhetos* (inexpressable) which is merely another way of expressing what the term *alogos* originally meant. It is also interesting to see how the term *alogos* gradually came back. First the term *rhetos* (rational) is created in contrast to *arrhetos*. Then the term *arrhetos* disappears; and Theaetetus, who developed the theory of irrationality further, reintroduced the term *alogos* but used it only for 'higher' irrationalities, for instance of the form $\sqrt{a\sqrt{b}}$, while he called the simple irrationalities of the form $\sqrt{a}$ *dynamei monon rhetoi* (literally: rational only in the square). Finally, when *logos* had become a technical term and the incongruity of the statement that two pairs of *alogoi* have the same *logos* was no longer felt, the Greek mathematicians returned to the old terminology and called all irrationals *alogos*.[85] The fact that Theaetetus, who died in 369 B.C., had already begun to return to the old terminology is a very strong confirmation of the view that the discovery of incommensurability must have been made long before, and that the term *logos* for ratio, from which

---

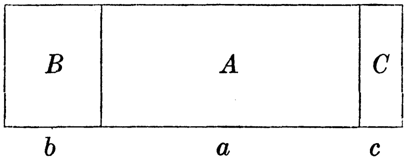
[84] The famous demonstrations of Hippocrates of Chius, who belonged to the same generation as Theodorus of Cyrene, clearly presuppose that the theory of proportions at his time had already been adapted to incommensurables. See F. Rudio, *Der Bericht des Simplicius über die Quadraturen des Antiphon und des Hippocrates* (Leipzig, Teubner, 1907), and infra p. 262.

[85] For details see my article on Theaitetos (see note 5), p. 1361 f.

alogos is derived, must certainly have been used by the Pythagoreans before the middle of the fifth century.

The extension of the theory of proportion to incommensurables required an entirely new concept of ratio and proportion and a new criterion to determine whether two pairs of magnitudes which are incommensurable with one another have the same *logos*. The early solution of this problem is most ingenious. Instead of making the result of the process of mutual subtraction the criterion of proportionality, namely the two sets of integers determined by measuring two commensurable magnitudes with the greatest common measure found by mutual subtraction, they used the character of the process of mutual subtraction itself as the criterion of proportionality. They established this criterion by giving a new definition of proportionality which made it applicable to commensurables and incommensurables alike. In literal translation this definition says: *magnitudes have the same logos if they have the same mutual subtraction.*[86] It is interesting to see that in this definition the term *logos* has lost its original meaning. The sense of the definition is, then, that two sets of magnitudes are in proportion if in each case the process of mutual subtraction, even if going on in infinitum, nevertheless can be proved always to go in the same direction.

To show this is especially easy in the case of the diameters and sides of all regular pentagons, since in this case, the diameter being cut in the so-called golden section, it is evident that the process will always go exactly one step in each direction. But if the practical applicability of the new definition had been limited to this case it would have been of little use for the further development of mathematical theory. The most important case in which it is very easy to prove on the basis of the new definition that two pairs of magnitudes are in proportion is the proposition that rectangles and (since parallelograms can very easily be converted into rectangles of the same area) parallelograms of the same altitude are in proportion with their bases.



For it is easy to see that if b can be subtracted 5 times from a, B can also be subtracted 5 times from A, and if the remainder c can be subtracted 8 times from b, so can C from B, and so forth in infinitum.[87] This proposition is the foundation of the famous theorems of Hippocrates of Chios.

---

[86] See Aristotle, *Topica*, 158 b, 32 ff.

[87] In literal translation the passage in Aristotle runs like this: 'It seems that in mathematical theory some propositions cannot easily be proved on account of the lack of a definition (or: as long as the proper definition is lacking), as for instance the fact that a straight line cutting an area parallel to its side cuts the area and its base in the same proportion (literally: similarly). But as soon as the definition has been found (the truth of)

Yet the usefulness of this new definition for the demonstration of geometrical propositions is still restricted to a rather limited field. The further expansion of the theory of proportions was made possible through the new and even more ingenious definition which was invented by Eudoxus of Cnidus and which runs as follows: *Magnitudes are said to be in the same logos, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and the third and any equimultiples whatever of the second and the fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*[88]

If one compares the discovery of incommensurability (assuming that it was made in the manner suggested above) with these extensions of the theory of proportions, it seems evident that the discovery of incommensurability was by far the easiest step. For once the Pythagoreans became interested in the pentagram and the regular pentagon, anyone might be struck by the fact that the diameters will always form a new regular pentagon in the centre; and if, furthermore, the general Pythagorean doctrine required the determination of the 'logos' of diameters and sides, all the rest followed very easily. Of the two new definitions of proportion, that of Eudoxus is perhaps the most ingenious inasmuch as it required the greatest effort in abstraction. But the older definition of proportion, by which the original concept of *logos* was replaced by a new one, which made it possible to apply the theory of proportion to incommensurables, was certainly by far the most important step in the development.

---

the proposition is at once manifest. For the areas and their bases have the same mutual subtraction; and this is the definition of proportion (*ho autos logos*)! It seems strange that O. Becker in an article published in 1933 (Quellen und Studien zur Geschichte der Mathematik, Abteilung B, vol. 2, pp. 311 ff.) was the first to give the correct interpretation of the expression 'have the same mutual subtraction' in the passage quoted, while Heath, for instance, still called the definition 'metaphysical,' and said that it was difficult to see how any mathematical facts could be derived from the definition.

O. Becker in a most excellent analysis has also proved that the greater part of the 10th book of Euclid's Elements which contains a very elaborate theory of irrationals can be proved by means of this definition, while some of the propositions specifically ascribed to Eudoxus cannot be proved on the basis of this definition and presuppose the new definition Euclid V, def. 5. Since the most important propositions of the 10th book of Euclid are ascribed to Theaetetus, Becker drew the obvious conclusion that Theaetetus worked with the old definition quoted by Aristotle.

This is undoubtedly correct. But his interpretation of the rest of the passage in Aristotle seems to require a slight modification. Though Becker has seen that the 'areas' in Aristotle are in fact parallellograms, or rather, rectangles, he believes that the proposition about rectangles was from the beginning proved by an elaborate process of reasoning, which required that several other propositions had been proved first (*op. cit.*, p. 322). This is certainly not what Aristotle indicates, when he says that the truth of the proposition is manifest as soon as the definition is found. For this expression shows clearly that originally a direct application of the definition to the figure given above was considered sufficient proof of the proposition. This is an interesting parallel to the first demonstration of incommensurability in the pentagon as suggested above.

[88] See Euclid, *Elements*, V, def. 5 and *Scholia in Euclid. Element.* V. 3 (Euclidis Opera. ed. I. L. Heiberg, vol. V, Leipzig, Teubner, 1889, p. 282.)

The fact that the development from the discovery of incommensurability to Eudoxus took this course has also chronological implications. Eudoxus was born in 400 and died in 347 B.C.[89] His last work, which he left uncompleted, was a large geographical work in many volumes.[90] He was also the author of the method of exhaustion, of the theorem that the volume of a cone is one-third of the volume of a cylinder with the same base and altitude,[91] and undoubtedly of other stereometric theorems which must have been used in the proof of that proposition. All this would have been impossible without the new definition of proportion invented by Eudoxus. He therefore must have created this definition comparatively early in his life, hardly later than 370. It would, then, be little less than micraculous if the first discovery of incommensurability had been made 'in the time of Archytas' who, since he was head of the government of Tarentum in 362, can hardly have been born before 430. It is certainly much easier to believe that the discovery was made in the middle of the fifth century, as ancient tradition claims.

But the solution of the chronological problem is of importance mainly because it makes it possible to acquire a deeper insight into the way in which the Greeks laid the foundations of the science of mathematics and into the special qualities which enabled them within an amazingly short time to make a discovery which their Babylonian and Egyptian predecessors with all their highly developed and complicated methods had not made in many centuries of mathematical studies.

COLUMBIA UNIVERSITY

---

[89] See K. von Fritz, 'Die Lebenszeit des Eudoxos von Knidos' in Philologus, 85 (1930). p. 478 ff.

[90] See F. Gisinger, *Die Erdbeschreibung des Eudoxos von Knidos*, p. 5 ff.

[91] See Archimedes, *Ep. ad Dositheum in De sphaera et Cylindro*, p. 4 Heiberg and *Ad Eratosth. Methodus*, p. 430 Heiberg.

The story of the discovery of incommensurability, revisited

D.H. Fowler

pp.221-235 of *Trends in the Historiography of Science*, ed K. Gavroglu, J. Christianidis, & E. Nicoliaidis, Boston: Kluwer, 1994, with some very minor variations

I take as my opening text the kind of thing my colleagues — certainly the mathematicians and often the historians of mathematics — might say about the beginnings of Greek mathematics. Something like this:

> The early Pythagoreans based their theory of proportion on commensurable magnitudes (or on the rational numbers, or on common fractions $^m/_n$), but their discovery of the phenomenon of incommensurability (or the irrationality of $\sqrt{2}$) showed that this was inadequate. This provoked problems in the foundation of mathematics that were not resolved before the discovery of the proportion theory that we find in *Elements* V.

You must, at some time or another, have heard, or perhaps even have said, something like that. I've certainly said it; I even got into *Punch* for saying it![1] But I shall try explain why I now disagree with everything in this line of interpretation. My space is limited so I will have to refer you for many of the crucial details in what follows to the thorough discussions I have cited or given in my book, *The Mathematics of Plato's Academy*; in fact, I hope this article will form the basis of the opening chapter of a sequel to this book.[2] I shall arrange my comments under various headings. First we have the matter of:

The nature of our evidence. Our evidence about Greek mathematics in general comes in very disparate forms, and almost all of it has been subject to an unknown amount of editing and interference. In particular our late sources — editions, compilations, and commentaries dating from the 2nd century AD onwards — are manifestly of very variable quality. So, in the first phase of my reconstruction, as represented by my book, I put it all to one side as far as possible, and ignore it.[3] (This is very drastic, and my hope is to consider some of this later evidence in the sequel.) Moreover, some of the relevant evidence in early sources, in particular Euclid and Plato, comes in homogeneous slabs which often fit rather awkwardly in the various versions of the received interpretation. In some measure to redress the balance after my radical approach to the late texts, I endeavour to follow the principle that, if any piece of such an early slab enters the reconstruction in a significant way, then the proposal should also engage with

---

[1] *Punch*, April 24, 1974. This magazine provided a humorous commentary on British life for 150 years until its closure in March 1992. In recent years, it ran a regular column 'Country Life' which explained itself as follows: "Not everything that happens in Britain gets into the national press. This feature presents some of the news that never made it." One reader sent in the following clipping: "The programme, which is about the development of number systems, will include an interview with David Fowler, of Warwick University, on the historical crises associated with the square root of two." This must ultimately have come from some Open University publicity about a TV programme I had helped make for their first History of Mathematics course, which had then been passed on by my university, picked up by the local newspaper, *The Leamington Spa Courier*, and submitted to *Punch* by a local reader. This programme was my first and reluctant venture into Greek mathematics, and I later disowned it, only permitting the Open University to continue broadcasting it if it also circulated a disclaimer by me to the students doing the course!

[2] Hereafter I shall refer to these as 'my book' and 'the sequel'.

[3] At the outset, I must admit that one piece of evidence of late provenance plays a crucial role in the reconstruction, namely the material on 'side and diagonal' numbers and lines, found in Theon of Smyrna, Iamblichus, and Proclus. See the discussion in my book, pp. 100-4, which however does not deal with the material in Iamblichus (ed. Pistelli, 91.3 - 93.6).

the whole of its context. For example, any important use of any aspect of the curriculum in Plato's *Republic* VII should also connect with the whole of this curriculum, especially since Plato insists on its unity; any significant application of a proposition in *Elements* II should eventually involve all 14 of them; anything about *Elements* XIII, the book which contains the construction of regular solids and a lot more, should say something about Book X, the classification of incommensurables, and perhaps also about Books IV and II.

Please note: I am not saying early evidence is all good, late evidence is all bad. I am saying that our evidence is a hotchpotch that we cannot sort out until later in the project, but the best evidence is likely to be found in the coherent and obscure chunks of early provenance (this is analogous to the principle *difficilior lectio* of textual criticism), so let us start from this material, taken all in one piece. It is a methodological principle, not a simple value judgement of the evidence.

Back now to my opening text. In fact, my principal objection to it is that it is founded entirely on late evidence and speculation, uncorroborated to a remarkable extent by any of our earlier sources, so it forms a very insubstantial base on which to start our reconstruction of Greek mathematics. I shall spell that out in more detail, and then propose an alternative starting point.

On Pythagoras and the early Pythagoreans, see, for example, W. Burkert, *Lore and Science in Ancient Pythagoreanism*. I subscribe fully to his general conclusion, that the only kind of scientific activities and discoveries we can attribute with confidence to the early Pythagoreans are some remarkable findings in music theory and acoustics, the most remarkable being that consonance is associated with small integers. Most, if not all, of the mathematical stories have the ring of later legendising. ("Pythagoras the mathematician finally died in 1962!", Reviel Netz.) In particular, consider:

The Pythagorean theory of proportion based on commensurable magnitudes. I know of no explicit evidence for this, early or late. From what I can work out, the thinking goes something like this: we, since medieval, times have learned at school about common fractions, that is $^m/_n$s, so that is what the Pythagoreans must have used, especially since we find these fractions later in Greek mathematics and Greek accounting. Occasionally this opinion gets expressed in print; see, for example, B.L. van der Waerden, *Science Awakening*, pp.49-50 & 115-6. He writes:

> It is probable that it was *calculation with fractions* which led to the setting up of *Elements* Book VII [which van de Waerden attributes to the early Pythagoreans]. Fractions do not occur within the official Greek mathematics before Archimedes but, in practice, commercial calculations had of course to use them ... .

Once again, I disagree with everything in this opinion. Fractions do occur in 'official mathematics', even, in a limited way, in the *Elements*: see the use of the word *meros*, plural *merê*, translated as 'part' and 'parts', especially in Book VII. For example, VII 37 & 38 talk of 'hononymous parts', of three & the third, four & the quarter, etc., and these *merê* are an ingredient of the way fractions are described in Greek. But I do not think there is any unambiguous evidence for *common* fractions, these $^m/_n$s, in any of our early texts including commercial calculations, and their apparent appearance in the late Byzantine copies which account for 99% or more of our evidence may be instead as scribal abbreviations. Our plentiful surviving explicit evidence is that Greek fractional practice was exactly the same as Egyptian practice: fractions were expressed as sums of the *merê*, as so-called unit fractions. The details of this argument are long, painful, and contentious because, while we have lots of different kinds of evidence, most of it is of the wrong sort or it comes from the wrong place or the wrong time. The details are given in Chapter 7 of my book and summarised in my article ‘Logistic and fractions in early Greek mathematics’. And if you find the full conclusions of my overall thesis too much to swallow, we need only a much weaker version of it here, that the Greek mathematicians before Plato and Eudoxus used not common fractions, but Egyptian fractions. I’ll return to fractions and arithmetic later.

Note that, here and elsewhere, ‘Greek’ as in ‘Greek mathematics’ simply means ‘written in Greek’. Almost all of our evidence has been transmitted via Egypt, and then via the whole eastern Mediterranean, and that, of course, complicates the argument. Also, concerning fractions and division, I think that Diophantus needs a separate discussion.

The topic of incommensurability. At the end of my book, I review all of the evidence on this topic for the thousand year period up to Proclus and, there and here, I refer you also to another such review in W.R. Knorr, *The Evolution of the Euclidean Elements,* Chapter 2. Here are some opinions from these reviews.

- In the first surviving explicit mention of incommensurability,[4] in Plato’s *Theaetetus* (147aff), the topic is handled confidently as a source of interesting mathematical research. Incidentally, as to the date of Theaetetus’ death, which is generally regarded as one fixed point, perhaps the only secure fixed point, in the shifting sands of the incommensurability issue, I also note that there is now one expert who dissents from the common view. H. Thesleff, specialist on pseudo-Pythagorean texts, writes: “I find it essential to note that historians of mathematics who take it for granted that Theaitetos was still alive in the 370s must be wrong. He made some important discoveries as a young man, and Plato and his friends were deeply impressed by this. But he is likely to have died in 390 BC.’[5]

- The celebrated passage in Plato’s *Laws* (817e ff) where Plato talks of “ignorance ... not worthy of human beings but pigs” is probably not referring to incommensurability in our sense here, but something else, very possibly the kind of techniques used in land measurement, where again things are not what mathematicians and historians of mathematics seem to assume they ought to be when, for instance, they parade stories from commentators about Egyptian land measurement as the origin of mathematics.

- Aristotle’s favourite mathematical illustration is “the incommensurability of the diagonal” as something all mathematicians know; but he never suggests that it is or ever was a disaster for any mathematical theory, even though, in closely related passages, especially in the *Metaphysics*, he is highly critical of the Pythagoreans. Curiously, Aristotle never specifies that he is talking of the diagonal of a square. Twice, both times in *Prior Analytics,* at 41a23ff & 50a35ff, he says something like: “from the assumption that the diagonal is commensurate, it follows that odd numbers are equal to evens”, but Aristotle gives no more details of what he means by this. I find completely convincing Knorr’s proposal (*Evolution,* pp.228-232) that the so-called Pythagorean proof, revolving around this statement, was tacked on to the end of *Elements* X with two clumsy proofs sometime after the time of Alexander of Aphrodisias in the 3rd century AD, in response to the needs of Aristotelian commentators, and that Alexander himself cobbled together the variant of this proof to be found in his commentary. However other natural mathematical explanations of this remark by Aristotle about odd and even numbers are possible, and our evidence for any interpretation of it so tenuous as to be unreliable as a basis for further reconstruction. I will come back again to Aristotle later.

---

[4] I here leave to one side the notorious ‘nuptial number’ at *Republic* 546c, with its talk of the ‘rational and irrational diameters of five’.

[5] H. Thesleff, Platonic chronology, on p. 18, n. 47.

- No surviving fragment or testimony of Eudoxus mentions incommensurability: the word-index to Laserre, *Die Fragmente des Eudoxos*, does not contain the words *(a)summetros*, *(ar)rhetos*, or *alogos*!. To us, it may seem blindingly obvious that the principal achievement of the Eudoxan theory of *Elements* V must have been to accommodate ratios of incommensurable magnitudes, but no ancient source says that, and I intend to include a chapter on Eudoxus in the sequel that tells this story differently. For the moment, I'll have to leave it at that, and add that you'll already find most of the ingredients in my book, worked out in some detail.

- Proclus never quotes anything from Eudemus on incommensurability, though he cites Eudemus by name several times, and also writes about incommensurability several times. The one apparent exception to this, the passage in the catalogue of geometers where Eudemus appears to refer to Pythagoras, is almost certainly an interpolation; and the remark there that Pythagoras discovered the "doctrine of proportionals" is a modern editorial emendation of the text, where all of our manuscripts unanimously have "the doctrine of the *alogos*".

- A proper discussion of this word *alogos* would take us on a very long excursion into *Elements* Book X; instead, let us here just look very briefly at Euclid: As far as I am aware, the only time the topic of incommensurability appears in Euclid's works is in *Elements,* Books X & XIII, which form our only coherent slab of early evidence on the topic, and a very massive, very coherent slab it is, in bulk and content well more than a quarter of the *Elements*. Any discussion of incommensurability should deal with Book X; but if you have never looked at Book X, I can promise you that you will find it difficult, so I recommend you to try to get hold of a pamphlet by C. M. Taisbak called *Coloured Quadrangles.* You may find difficult too — I mean getting hold of Taisbak's pamphlet! — so I have given a version of it in Chapter 5 my book, and subsequently written an improved version of this: 'An invitation to read Book X of Euclid's *Elements'*. As to the role of Books X and XIII in my reconstruction of the incommensurability story, I must refer you to the rest of Chapter 5 of my book.

- The source of most of the stories about Pythagoras, Pythagoreanism, and incommensurability is the book *On the Pythagorean Life* by Iamblichus, so perhaps it is worth quoting the relevant passages in full:[6]

> §18 (88) ... As for Hippasos, he was indeed a Pythagorean, but because he was the first to make public the sphere constructed from twelve pentagons he was lost at sea for his impiety: he got the reputation of having discovered it, but it all came from 'that man' — that is what they call Pythagoras: they do not use his name.
>
> §34 (246) ... The first man to reveal the nature of commensurability and incommensurability[7] to those unworthy to share his teachings was so much detested, they say, that not only was he excluded from their common life and meals, but they built him a tomb as if their former companion had left human life behind. (247) Some say the supernatural power took revenge on those who published Pythagoras' teachings. The man who revealed the construction of the 'twenty-angled shape' was drowned at sea like a blasphemer. (He told

---

[6] These quotations are taken from a new English translation of Iamblichus by G. Clark, *Iamblichus: On the Pythagorean Life,* Liverpool University Press, 1989.

[7] The translation here has "symmetry and asymmetry", but surely (in)commensurability is the appropriate translation here of *(a)saummetros,* as in the phrase "irrationality *(alogos)* and incommensurability" at end of the passage. The later "twenty-angled shape" is *eikosagonon,* a non-standard name (perhaps found only here in Iamblichus) for *dodekaedron.*

> how to make a dodecahedron, one of the ‘five solid figures’, into a sphere.) Some say this fate befell the man who told about irrationality and incommensurability.

This farrago of mutually inconsistent stories, which appear for the first time in a source of doubtful reliability and relevance dating from some nine centuries after the time of Pythagoras, is the main evidential base for much of what has been written about the discovery of incommensurability! I add the slightly perplexed comment that the most recent and authoritative study of the role of mathematics in neo-Pythagorism, D.J. O’Meara’s book *Pythagoras Revived, Mathematics and Philosophy in Late Antiquity,* does not even seem to mention incommensurability.

The foundation crisis. Again I find Freudenthal, Knorr,[8] and other writers very convincing when they argue that, far from being a period of crisis and confusion, the early fourth century was an extraordinary period of creativity, especially in Plato’s circle; we have no historical evidence for any of the postulated difficulties of a ‘foundation crisis’. But I want to go further and explore the possibility that the discovery was an incidental event in the early development of mathematics. So let us now look at the evidence concerning the effects of the discovery. This will involve tracking over some of the same material again.

The implications of the discovery of incommensurability. Just what precisely are the supposed problems raised by the discovery of incommensurability? As far as I know, no Greek text, early or late, tells us clearly of the mathematical difficulties raised by the phenomenon. Aristotle wrote of the innocent's surprise, and the way it then gives way to a more informed appreciation:

> All men begin ... by wondering that the matter is so (as in the case of automatic marionettes or the solstices or the incommensurability of the diagonal; for it seems wonderful to all men who have not yet perceived the explanation that there is a thing which cannot be measured even by the smallest unit). But we must end in the contrary and, according to the proverb, the better state, as is the case in these instances when men learn the cause; for there is nothing which would surprise a geometer so much as if the diagonal turned out to be measurable (*Metaphysics* 983a12-20).

Pappus’ *Commentary on Book X of Euclid’s Elements,* I.2, writes, much later, of how thereafter:

> the soul ... wanders hither and thither on the sea of non–identity ... immersed in the storm of the coming–to–be and the passing–away, where there is no standard of measurement

from which passage I shall grasp, below, on the only substantial straw, the last four words: "there is no standard of measurement". There is the similar Scholium 1 to Book X, quoted in part in the Introduction to Book X in Heath's translation of the *Elements*, where you will find a long discussion that is remarkable for its lack of any hard evidence.[9] And Proclus, in his *Commentary on the First Book of Euclid’s Elements,* 60, writes:

> The statement that every ratio is expressible (*rhetôs*) belongs to arithmetic

---

[8] See Freudenthal, ‘Y avait-il une crise de fondements...’ and Knorr, *Evolution,* 306 -12.

[9] Contrast this with the sparse sections in Thomas, *Selections* i pp. 110-11 & 214-17, on ‘The Irrational’, where this and a passage from Aristotle are the only texts excerpted.

> only, and not to geometry, for geometry contains inexpressible ratios (*arrhetos logos*)

but this is not, very much not, the terminology of *Elements* X, which is built on a completely different meaning for expressible incommensurable lines and ratios.

So, prompted by Aristotle, let us try to "perceive the explanation" and "learn the cause" of incommensurability. But first, may I persist a bit longer with a variation on my question at the beginning of this section:

> What precisely, to modern commentators, are the supposed difficulties revealed by incommensurability?

A whole range of possibilities now seems to be on offer. With some overlap, there are the following:

• The Proclus objection: Incommensurable ratios cannot be expressed in (or by) numbers. Surely that is nonsense! We express them in numbers, and much of mathematics is, ultimately, numbers. Early Greek mathematicians could have expressed them in numbers. We have no explicit evidence that they did this, but I am arguing for a speculative interpretation in which they were expressed in numbers. My book is full of examples.

• Incommensurable ratios could not be fitted within the pre-Eudoxan style and scope of mathematics. That is false! The definition for lines that $a:b::c:d$ means rectangle (a,d) = rectangle (b,c) uses only believed-to-be early ingredients in a believed-to-be early way, and will handle most of what we need for the *Elements*, perhaps everything except for compounding, on which Euclid's own treatment is strange and unsatisfactory. But we have no evidence that this was an early definition.

• Incommensurability showed the inadequacy of the Pythagorean doctrine that "all is number". Therefore, some add, it had to be concealed. Curiously, Aristotle, our principal early witness, from whom we learn of this "all is number", never advances this criticism, even though all of this material comes together in the *Metaphysics*. We find there lots of mentions of incommensurability, we find summaries and harsh criticisms of Pythagorean philosophy, but we don't find this objection. Zhmud has recently even proposed that this formulae "all is number" was Aristotle's invention.[10] And, I would add, incommensurability need not show the inadequacy of the doctrine that "all is number", *pace* Proclus; indeed, in my reconstruction, the behaviour of the anthyphairetic ratios of $\sqrt{n}:\sqrt{m}$ might even reinforce such a doctrine, and give a mathematical explanation of the 'expressible' lines that underlie *Elements* X and XIII.

• The discovery of incommensurability showed that the [Babylonian?] arithmetical basis of geometry was inadequate, so geometry had to be reformulated purely geometrically, for example, as in *Elements* II. We have no evidence of this supposed early, possibly Babylonian, arithmetical basis of early Greek mathematics, so this is pure speculation. Also, there are other explanations of the role of *Elements* II. I shall return to this issue later, when I discuss arithmetisation.

• Incommensurable ratios can only be approximated, while Greek mathematics aimed for precision. This assertion runs counter to our evidence: Archimedes is interested in

---

[10] L.Y. Zhmud, '"All is number"? "Basic doctrine" of Pythagoreanism reconsidered'.

approximation. Also Aristarchus and then, later, Hypsicles, Hero, Ptolemy, ... Moreover the fraction 1/7 can only be approximated in sexagesimal arithmetic, and the fraction 1/3 can only be approximated in the Greek system of land measurement which, by convention, only uses the parts 2', 4', 8', 16',... . So Greek mathematics was involved in approximation and the necessity of approximating simple numerical ratios was a well-known phenomenon.

- The discovery put into question the basic idea of ratio. I prefer to reformulate this, since no early text seems to express this concern: hints of it seem to surface first in later commentators like Iamblichus and Pappus, and then grow thereafter. So I shall change the question once again and ask instead:

> Why do none of the early testimonies seem concerned with the manifest difficulties that incommensurability poses to our way of defining ratio?

This emphasises that the difficulties may be *our* difficulties, not necessarily theirs; they may have had different ways of thinking about ratio. I finish by developing this theme a bit.

Incommensurability does present a problem to arithmetised geometry, though ultimately I think it turns out that the problem was already lurking there even for commensurable manipulations. Let me try to explain. Arithmetised geometry is how we tend to think of geometry today: a line has a length, a number; a rectangle has an area, again a number which is equal to the product of the lengths of its sides; ratios are defined arithmetically, as quotients of numbers; and so on. So the geometry becomes translated into the arithmetical manipulation of numbers — addition, subtraction, multiplication, division, taking roots, etc. — and then this arithmetic is later abstracted into algebra. For example, the so-called Pythagoras' theorem becomes $a^2 + b^2 = c^2$, where the usual interpretation of this statement involves the lengths (i.e. numbers) a, b, and c of the sides of the triangle. It then seems that the numbers associated with things like $\sqrt{2}$ are rather complicated, so complicated that filling in all of these irrational numbers properly could not be done before the middle of the nineteenth century. Dedekind, the first, tells us that he succeeded on 24 November 1858; it was a Wednesday.

But read Dedekind carefully, and you will see that there are already serious problems with arithmetic. I go yet further than this, and argue that there need be no real problems in defining the set of numbers, for example, as decimal sequences, or sexagesimal sequences, or anthyphairetic sequences, or my astronomical sequences, or other such descriptions. However, a precise description of the arithmetic, even the of the arithmetic of the rational numbers when they are not being conceived as common functions, is intractable.[11] Hence the crucial role of my belief that early Greeks did not use common fractions, so they would not think of ratios in any way like our rational numbers, and so would not think that arithmetic was a natural and obvious basis on which to build their mathematics.

To pursue this theme further, early Greek geometry seems to me to be not arithmetised to a remarkable extent,[12] but later Greek mathematics is arithmetised. There are two different

---

[11] For discussions and examples, see my ‘400 years of decimal fractions’, ‘400.25 years of decimal fractions’, and 'Dedekind's theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$'.

[12] The only arithmetised passage I know, anywhere up to Archimedes and beyond, is in Plato's *Meno*, 82c ff, where Socrates says to the slaveboy: "Now if this side is two feet long, and this side the same, how many feet will the whole be"; and the passage continues in this arithmetised vein for the slaveboy's first two attempts. (I thank Wilbur Knorr for pointing this out to me in January 1991 in a train somewhere between Verona and Venice; I had used this passage for the introduction to my book without appreciating this feature!) The switch to geometry is then indicated by Socrates when he tells the slaveboy: "If you don't want to count it up, just show me on the diagram" (84a). And I think one can explain this singular exception by observing that the the

traditions in this later arithmetisation. The first is the astronomical one, using Babylonian sexagesimal numbers, which is not attested in Greece before Hypsicles and Hipparchus in the 2nd century BC. It may have been transmitted much earlier, but that is pure speculation, as was remarked earlier. The second is the Graeco-Egyptian unit fraction tradition which we find in our earliest testimonies like the Hibeh Papyrus, then later in the Heronian Corpus, and in Ptolemy alongside the sexagesimal astronomical calculations. The arithmetic of both poses theoretical problems, and both are spectacularly absent from the geometry of the *Elements*: they are incompatible with the spirit of the proportion theory of Book V and remote from the spirit of the treatment of incommensurability of Book X. No wonder commentators from antiquity onwards, who seem to work in a vaguely arithmetised geometry, have a hard time, and no wonder there seems to be some confusion.

So let us try to purge our mind of this arithmetised geometry. (I did not find this easy and it took me some years spent with some good alternatives.) One problem we then face is defining ratio or proportion. (Note that this problem is not now directly concerned with incommensurability.) I have already pointed out that we have no real difficulty in fitting proportionality into the pre-Eudoxan style of mathematics. But our only evidence on how early Greek mathematicians actually handled ratio or proportion is the celebrated passage in Aristotle's *Topics* 158b29ff on *antanairesis/arthyphairesis* which suggests the use of the so-called Euclidean algorithm:

> Given two numbers or two lines (or, with a bit of technique at our disposal, two more complicated geometrical objects), then see:
> how many times the second line can be subtracted from the first line;
> how many times the remainder can be subtracted from the second line;
> how many times the next remainder can be subtracted from this remainder;
> etc.
> and this gives a string of numbers that characterise the relationship of size between the two things.

For example, the ratio of 60 to 26 will be twice, three-times, four-times exactly. Do the process for two numbers, and you will quickly see that it must stop after a finite number of steps, because if not "an infinite series of numbers will [arise], each of which is less than the other, which is impossible in numbers" (as formulated in *Elements* VII 31). Now do it with two lines. Early Greek geometers seem to take little notice of the philosopher's problems, and their lines can be chopped up indefinitely, so the possibility of the anthyphairesis going or indefinitely presents itself. If you are a geometer, possessed of a bit of technique, and this question poses itself, you will soon find examples of it happening. Thus, as Aristotle might be saying, we could "learn the cause" and "perceive the explanation" of incommensurability in this way of thinking about the ratios of lines. But if, as Pappus might be saying, we involve geometry in "standards of measurement", which I take here as a hint of arithmetisation, then we encounter problems, as I have tried to explain.

Here then are some illustrations. First, consider the problem of ‘the diagonal and the side’: draw any regular polygon, and evaluate the anthyphairetic ratio of one of its diagonals and its side. The easiest and best known example is the pentagon, so I leave that for the reader, and will consider here the square. We easily see that the side S of a square goes once, but not twice, into its diagonal D (this follows from Socrates’ comments at Plato’s *Meno*, 82a-85d), and so the ratio(diagonal to side) is once, followed by the ratio(side to diagonal minus side).

---

slaveboy is not a mathematician — that is the whole point of the episode. Note that an arithmetisation of aspects of everyday life occurs when a barter economy gives way to the use of money, which seems to have happened in Greece by the 7th century.

We are now faced with the evaluation of this ratio of the side to diagonal minus side, and some may feel that, like Meno at 84a, that we have made little progress: “It’s no use Socrates, I just don’t know”. But, just as Socrates unblocks that impasse by conjuring a clever figure out of thin air, so I here draw Figure 1: Starting from the small diagonally placed square in the left-hand corner, with side s and diagonal d, we construct a larger square whose side S is s+d, and check that its diagonal D is 2s+d.[13] (Note how the symbols, here and below, are pure shorthand for lines and contain no hint of arithmetisation.)

With one eye on Figure 1, and with the insight that the ratios are unaffected by the scale or orientation of the figures involved, we take up our problem again, and see that the ratio(big side to big diagonal minus side) is the same as the ratio(little side plus diagonal to little side); and we can now evaluate this as twice, followed by the ratio(little side to little diagonal minus side) which is — scale and orientation aside — what we just started from. Hence the ratio(side to diagonal minus side) is twice, twice, twice, twice, continuing thus indefinitely, and so the ratio(diagonal to side of a square) is once, twice, twice, twice, ...

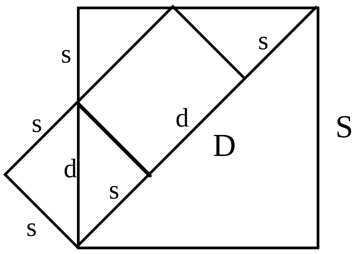


Figure 1

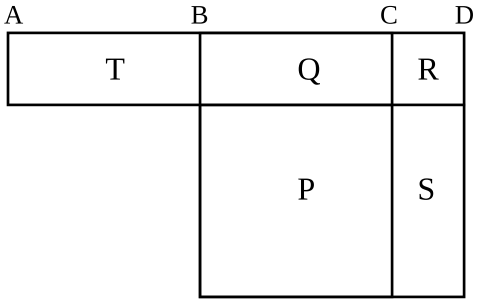


Figure 2

Let me evaluate this same ratio another way; or, to be more exact, let me give another proof that ratio(diagonal plus side to side) is twice, twice, twice, twice, ... Start with a square P and, by adding on a gnomon Q+R+S = P, as in Figure 2, construct a larger square of size 2P, whose side will therefore be the diagonal of the original square; and then append a rectangle T equal to Q, as shown. Then ratio(diagonal plus side to side) will be ratio(AD to BC), which we immediately see is twice, followed by ratio(BC to CD). The required proof will now follow if we can show that ratio(BC to CD) is equal to ratio(AD to BC); or equivalently, by a manipulation of geometrical proportions (see *Elements* VI 16; this manipulation was alluded to earlier), that rectangle(AD, CD) = square(BC), that is T+Q+R = P; but this underlies our construction, since T = Q = S. Q.E.D.

The case of the ratio of the longer diagonal to side of a hexagon gives rise to the ratio $\sqrt{3}$ to 1, where $\sqrt{3}$ denotes the side of the 3-fold square (which can be constructed, for example, using *Elements* II 14); and this example can be generalised to the investigation of the ratios like $\sqrt{n}$ to $\sqrt{m}$ — what I propose to call the problem of ‘the dimensions of squares’. We first use numerical techniques very close to those found in *Elements* VII to explore and conjecture what the answer might be. The second kind of proof above then deals with the simpler cases; a comparison of the mechanisms of Figure 2 and *Elements* II 11, gives us insight into the particular example of the ratios n-times, n-times, n-times, ... ; and yet more heuristic exploration, followed by a generalisation of Figure 1, gives a complete solution of this remarkable problem and a new interpretation of the whole of *Elements* II.

---

[13] This is my proposed interpretation of the figure being described in the texts on side and diameter lines; see note 3, above.

An analogous problem of ‘the dimension of cubes’ beckons, but proves to be of such redoubtable difficulty that in fact it remains unsolved today; Plato’s remarks at *Republic* 528b-c are still perfectly applicable! We can also explore related problems such as ‘the circumdiameter and side’ of polygons or polyhedra and see where they lead: they can provide explanations of the roles of *Elements* IV, X, and XIII. And the similar problem of ‘the perimeter and the diameter’ might be what lay behind Archimedes’ original calculation in his *Measurement of a Circle.*

Developing these ideas takes us into a different world. The ingredients are all early Greek, but they are fitted together to create a completely different picture. It is mathematically appealing, amazingly coherent (too coherent, in fact, for my comfort!), and consonant with a quite extraordinary breadth of our evidence. And the topic with which I started, the simple discovery of incommensurability, plays no significant part in it, which is why my book contained no discussion of it beyond the bald and sceptical catalogue of our evidence at the end of its penultimate chapter.

I'm constantly tinkering with the details of my book, so I shall finish with one final little addition, in the first of my dialogues, where my slaveboy discovers this anthyphairetic definition of ratio under Socrates' prompting. They start doing it on numbers (heaps of stones, in fact) and the slaveboy realises that the process must terminate. Then Socrates introduces the possibility of doing it with lines. At this point, at the end of B_{36} on p. 28, add: ‘I wonder if it now can go on for ever’. That's my slaveboy realising one of the causes and explanations of incommensurability, and it is just a passing remark on the way to discovering much more remarkable things alluded to in S_{37} - S_{43}, alongside which this simple fact of incommensurability fades into insignificance.

References

W. Burkert, *Lore and Science in Ancient Pythagoreanism,* Cambridge, Massachusetts: Harvard University Press, 1972; translation by E.L. Minar of *Weisheit und Wissenschaft,* Nurenberg, 1962.

R. Dedekind, *Stetigkeit und die irrationale Zahlen* (1872) &,*Was Sind und was sollen die Zahlen* (1888); repr. pp.315-391 of *Gesammelte mathematische Werke,* vol. 3, Braunschweig, 1932; tr. W.W. Beman in *Essays on the Theory of Numbers,* Chicago: Open Court, 1901.

D.H, Fowler, 400 years of decimal fractions, *Mathematics Teaching* 110 (1985) 20-21, & 400.25 years of decimal fractions, *ibid.* 111 (1985) 30-31.

D.H, Fowler, *The Mathematics of Plato's Academy: A New Reconstruction,* Oxford University Press, 1987.

D.H. Fowler, Logistic and fractions in Greek mathematics: a new interpretation, pp.133-147 in P. Benoit, K. Chemla, & J. Ritter, *Histoire de Fractions, Fraction d'Histoire,* Stuttgart: Birkhaüser, 1992.

D.H. Fowler, Dedekind's theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, *The American Mathematical Monthly* 99 (1992) 725-733.

D.H, Fowler, An invitation to read Book X of Euclid's *Elements, Historia Mathematica* 19 (1992) 233-264.

H. Freudenthal, Y avait-il une crise de fondements.des mathématiques dans l'antiquité, *Bulletin de la Société Mathématique de Belgique* 18 (1966) 43-55.

T.L. Heath, *The Thirteen Books of Euclid'S Elements,* 2nd. ed., Cambridge University Press, 1926.

W.R. Knorr, *The Evolution of the Euclidean Elements: A Study of the Theory Incommensurable Magnitudes and Its Significance for Early Greek Geometry,* Dordrecht: Reidel, 1975.

D.J. O'Meara, *Pythagoras Revived, Mathematics and Philosophy in Late Antiquity,* Oxford University Press, 1989.

C.M. Taisbak, *Coloured Quadrangles: A Guide to the Tenth Book of Euclid's Elements,* Copenhagen: Museum Tusculanum Press, 1982.

H. Thesleff, Platonic chronology, *Phonesis* 34 (1989) 1-26.

I. Thomas [=Bulmer-Thomas], *Selections Illustrating the History of Greek Mathematics,* vol i, London/New York: Loeb Classical Library, 1939.

B.L. van der Waerden, *Science Awakening,* Groningen: Noordhoff, 1954; translation by A. Dresden of *Ontwakende Wetenschap,* 1950, Groningen: Noordhoff.

L.Y. Zhmud, "All is number"? "Basic doctrine" of Pythagoreanism reconsidered, *Phronesis* 34 (1989) 270-92.