Academic Journal of Science,
CD-ROM. ISSN: 2165-6282 :: 05(01):373–382 (2016)

A LEAST SQUARES RATİO (LSR) APPROACH TO FUZZY LİNEAR REGRESSİON ANALYSİS

Murat Yazici

JForce Information Technologies Inc., Turkey

The Ordinary Least Squares (OLS) approach to Fuzzy Linear Regression Analysis which includes a double linear adaptive fuzzy regression model is one of the used methods for fitting a regression model and forecasting. The double linear adaptive fuzzy regression model based on two linear models: a core regression model and a spread regression model. It aims to minimize the Euclidean distance between the observed dependent fuzzy values and the obtained dependent fuzzy values Y_i^*. This paper includes a regression method called Least Squares Ratio (LSR) approach to fuzzy linear regression, and comparison of OLS and LSR according to Mean Absolute Error (MAE). In this study, symmetric triangular fuzzy numbers are used, the observed symmetric triangular fuzzy numbers are shown as $Y_i \equiv (c_i, r_i)$, c_i and r_i the obtained symmetric triangular fuzzy numbers are shown as $Y_i^* \equiv (c_i^*, r_i^*)$. c_i and r_i indicates the observed centers and spreads, c_i^* and r_i^* indicates the obtained centers and spreads, respectively.

Keywords: Estimation, Least Squares Ratio (LSR) method, Ordinary Least Squares (OLS) method, Fuzzy linear regression analysis.

Introduction

Vague or fuzzy data and application in several fields, such as psychometry, reliability, marketing, quality control, ballistics, ergonomy, image recognition, artificial intelligence, etc. A typical problem where vague data arise is that of assigning numbers to subjective perceptions or to linguistic variables (such as "enough", "good", "sufficiently", etc.). In fact, there are many cases where observations cannot be known or quantified exactly, and, thus, we can only provide an approximate description of them, or intervals to enclose them. For instance, "in measuring the influence of character size on the reading ability of a video display terminal […] the reading ability of the subject, which is essentially the experimental output, depends on his/her eyesight, age, the environment, individual responses, and even how tired is the individual. Some of these factors cannot be described accurately and […] the best description of these kinds of output is that they are fuzzy outputs" [8] [9].

Fuzzy linear regression was first introduced by Tanaka et al. (1980, 1982). Several developments followed these papers, such as those studied by Chang and Lee (1994 a-c, 1996), Diamond (1990), Diamond and Kloeden (1994), Kacprzyk and Fedrizzi (1992), Kim et al. (1996), Moskowitz and Kim (1993), Peters (1994), Redden and Woodall (1994, 1996), Savic and Pedrycz (1991), Tanaka (1987), Tanaka and Watada (1988), Tanaka and Ishibuchi (1991), Tanaka et al. (1995), Xizhao and Minghu (1992). Also, several applications and methods have been proposed: for instance, see Chang et al. (1996), Lee (1996), McCauley and Wang (1997), Kim and Bishu (1998), Wang and Tsaur (2000), Kao and Chyu

(2002), Nasrabadi and Nasrabadi (2004), Bargiela et al. (2007), González-Rodríguez et al. (2009), Lu and Wang (2009), Sener and Karsak (2011), Mashinchi and Orgun (2011), Taheri (2012), Kazemi and Hosseinzadeh (2012), Parvathi et al. (2013) and the others.

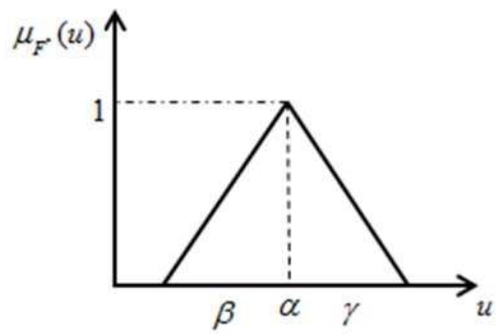

Figure 1. Triangular membership function.

Following Dubois and Prade (1980) and Zimmermann (1985), a fuzzy number may be defined as $F = (\alpha, \beta, \gamma)_{LR}$, where α denotes the *center* (or *mode*), β and γ are the left *spread* (or *width*) and right spread, respectively, L and R denote the left and right *shape functions*. When $\beta = \gamma$ and the shape functions are specular (w.r.t. the center), we have a *symmetrical* fuzzy number, denoted by $F^* = (\alpha, \beta)$. It is common to define a symmetrical fuzzy number by using a *triangular membership function* also shown in Fig. 1 [8]:

$$\mu_{F^*}(u) = \begin{cases} 1 - \dfrac{|\alpha - u|}{\beta} & \text{if } \alpha - \beta \leq u \leq \alpha + \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In the following sections, Least Squares Ratio (LSR) Regression and the OLS approach to fuzzy linear regression analysis developed by D'Urso and Gastaldi are explained. Then, the regression parameters of their double linear adaptive fuzzy regression model are estimated by using the LSR method. And, OLS and LSR's performances are compared by Mean Absolute Error (MAE).

Least Squares Ratio Regression

The Least Squares Ratio (LSR) method is one of the forecasting techniques in regression analysis. LSR aims to estimate observed values with zero error ($Y = \hat{Y}$, or $Y - \hat{Y} = 0$). It starts with the same goal $Y = \hat{Y}$ as in Ordinary Least Squares. However, it proceeds by dividing through by Y and so $\hat{Y}/Y = 1$ is obtained under an assumption of $Y \neq 0$. Hence, it is obvious that equations $1 - (\hat{Y}/Y) = 0$ and $(Y - \hat{Y})/Y = 0$ are raised by basic mathematical operations. This final equation is taken into account as the origin of the LSR which minimizes the sum of $[(Y - \hat{Y})/Y]^2$. Consequently, the aim of LSR can be written mathematically as follows [7]:

$$\min_{\beta} \sum_{i=1}^{n} \left(\frac{Y - \hat{Y}}{Y} \right)^2 . \tag{2.1}$$

The matrix representation of the regression model is as follows;

$$Y=\beta X+e \tag{2.2}$$

where Y is an n x 1-vector of observed values; X is an n x p-vector of the values of dependent variables; n is the number of observations; p is the number of unknown parameters, β is the p x 1-vector of regression coefficients; e is an n x 1-vector of error values.

Formula 2.1 can also be written as in formula 2.3, by using Eq. 2.2:

$$\min_{\hat{\beta}_{LSR}} \sum_{i=1}^{n} \left(\frac{Y - \hat{\beta}_{lsr} X}{Y} \right)^2 \tag{2.3}$$

If *rank(X)* is equal to p, the formula for estimating β appears as in Eq. 2.4 [7]:

$$\hat{\beta}_{lsr} = \left[\left(\frac{X}{Y} \right)' \left(\frac{X}{Y} \right) \right]^{-1} \left(\frac{X}{Y^2} \right)' Y \tag{2.4}$$

The matrix X/Y is obtained by dividing the values x_{ij} by y_i for $j = 1, 2, ..., p$ and X/Y^2 is computed by dividing the values x_{ij} by y_i^2 for $j = 1, 2, ..., p$.

The OLS Approach to Fuzzy Linear Regression Analysis

The double linear adaptive fuzzy regression model defined D'Urso and Gastaldi, 2000 is as follows:

$$\mathbf{c} = \mathbf{c}^* + \varepsilon_c \text{ where } \mathbf{c}^* = \mathbf{Xa} \tag{3.1}$$

$$\mathbf{r} = \mathbf{r}^* + \varepsilon_r \text{ where } \mathbf{r}^* = \mathbf{c}^* b + \mathbf{1} d \tag{3.2}$$

where $\mathbf{X}$ is a n x $(k+1)$-matrix containing the input variables; $\mathbf{a}$ is a column $(k+1)$-vector containing the regression parameters of the first model called core regression model; $\mathbf{c}$ and $\mathbf{c}^*$ are the vector of the observed centers and the vector of the obtained centers, respectively, both having dimension n x 1; $\mathbf{r}$ and $\mathbf{r}^*$ are the vector of the observed spreads and the vector of the obtained spreads, respectively, both having dimension n x 1; $\mathbf{1}$ is a n x 1-vector of all 1's, b and d are regression parameters for the second regression model called spread regression model.

The OLS approach to Fuzzy Linear Regression Analysis aims to minimize the euclidean distance between the observed dependent fuzzy values Y_i and the obtained dependent fuzzy values Y_i^*. The Euclidean distance between two symmetrical fuzzy numbers $Y_i \equiv (c_i, r_i)$, $Y_i^* \equiv (c_i^*, r_i^*)$ is as follows:

$$\delta_i \equiv \delta\left(Y_i, Y_i^*\right) = \sqrt{\left(c_i - c_i^*\right)^2 + \left(r_i - r_i^*\right)^2}\,. \tag{3.3}$$

The regression parameters are estimated by minimizing the following sum of square errors

$$\varphi\left(\mathbf{a}, b, d\right) \equiv \sum_{i=1}^{n} \delta_i^2 = \left(\mathbf{c} - \mathbf{c}^*\right)' \left(\mathbf{c} - \mathbf{c}^*\right) + \left(\mathbf{r} - \mathbf{r}^*\right)' \left(\mathbf{r} - \mathbf{r}^*\right). \tag{3.4}$$

By solving problem (3.4) we have the following iterative least-squares estimates [8]:

$$\mathbf{a} = \frac{1}{(1+b^2)} ((\mathbf{X'X})^{-1} \mathbf{X'}(\mathbf{c} + \mathbf{r}b - \mathbf{1}bd)), \tag{3.5}$$

$$b = (\mathbf{a'X'Xa})^{-1}(\mathbf{r'Xa} - \mathbf{a'X'1}d)\,, \tag{3.6}$$

$$d = \frac{1}{n}(\mathbf{r'1} - \mathbf{a'X'1}b)\,. \tag{3.7}$$

The Proposed LSR Approach to The Doubly Linear Adaptive Fuzzy Regression Model

The Proposed LSR Approach to The Doubly Linear Adaptive Fuzzy Regression Model aims to minimize the euclidean distance between the observed dependent fuzzy values Y_i / Y_i and the obtained dependent fuzzy values Y_i^* / Y_i.

The proposed LSR approach is defined as follows;

$$\mathbf{1} = \left(\frac{c^*}{c}\right) + \mathbf{r}\varepsilon_{1c} \quad \text{where} \quad \left(\frac{c^*}{c}\right) = \left(\frac{\mathrm{X}}{\mathrm{c}}\right)\mathbf{a}\,, \tag{4.1}$$

$$\mathbf{1} = \left(\frac{r^*}{r}\right) + \mathbf{r}\varepsilon_{1r} \quad \text{where} \quad \left(\frac{r^*}{r}\right) = \left(\frac{c^*}{c}\right)b + \left(\frac{1}{r}\right)d\,, \tag{4.2}$$

where X/c, computed by dividing the values x_{ij} by c_i for $i = 1,2,...,n$ and $j = 1,2,...,p$, is a n x $(k+1)$-matrix containing the input ratio variables; $\mathbf{a}$ is a column $(k+1)$-vector containing the regression parameters of the first model called core regression model; c^*/c is the vector of the obtained center ratios, having dimension n x 1; $\mathbf{r}$ is the vector of the observed spread values, having dimension n x 1; r^*/r is the vector of the obtained spread ratios, having dimension n x 1; $1/r$ is computed by dividing the values 1 by r_i for $i = 1,2,...,n$, having dimension n x 1; $\mathbf{1}$ is a n x 1-vector of all 1's, b and d are regression parameters for the second regression model called spread regression model.

The proposed LSR approach to the double linear adaptive fuzzy regression model aims to minimize the euclidean distance between the observed ratio values Y_i / Y_i and the obtained ratio values Y_i^* / Y_i. The Euclidean distance between $\frac{Y_i}{Y_i} \equiv (\frac{c_i}{c_i}, \frac{r_i}{r_i}) = (1,1)$ and $\frac{Y_i^*}{Y_i} \equiv (\frac{c_i^*}{c_i}, \frac{r_i^*}{r_i})$ is as follows:

$$\delta_i \equiv \delta\left(\frac{Y_i}{Y_i}, \frac{Y_i^*}{Y_i}\right) = \delta\left(1, \frac{Y_i^*}{Y_i}\right)\sqrt{\left(1 - \frac{c_i^*}{c_i}\right)^2 + \left(1 - \frac{r_i^*}{r_i}\right)^2} \tag{4.3}$$

The regression parameters are estimated by minimizing the following sum of square ratios (with matrix notation):

$$\varphi(\mathbf{a}, b, d) \equiv \sum_{i=1}^{n} \delta_i^2 = \left(\mathbf{1} - \left(\frac{c^*}{c}\right)\right)'\left(\mathbf{1} - \left(\frac{c^*}{c}\right)\right) + \left(\mathbf{1} - \left(\frac{r^*}{r}\right)\right)'\left(\mathbf{1} - \left(\frac{r^*}{r}\right)\right) \tag{4.4}$$

Theorem 1 (*least squares ratio iterative solutions*). Recursive solution for the problem of the least squares ratio estimation with fuzzy data is given by

$$\mathbf{a} = \frac{1}{1+b^2}\left(\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\right)^{-1}\left(\left(\frac{X}{c}\right)'\left(\mathbf{1} + \mathbf{1}b - \left(\frac{1}{r}\right)bd\right)\right), \tag{4.5}$$

$$b=\left(\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}\right)^{-1}\left(\mathbf{1}'\left(\frac{X}{c}\right)\mathbf{a}-\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)d\right), \tag{4.6}$$

$$d=\left(\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)\right)^{-1}\left(\mathbf{1}'\left(\frac{1}{r}\right)-\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)b\right). \tag{4.7}$$

$$\varphi(\mathbf{a},b,d)=\left(\mathbf{1}-\left(\frac{c^*}{c}\right)\right)'\left(\mathbf{1}-\left(\frac{c^*}{c}\right)\right)+\left(\mathbf{1}-\left(\frac{r^*}{r}\right)\right)'\left(\mathbf{1}-\left(\frac{r^*}{r}\right)\right)$$

$$=\left(\mathbf{1}-\left(\frac{X}{c}\right)\mathbf{a}\right)'\left(\mathbf{1}-\left(\frac{X}{c}\right)\mathbf{a}\right)+\left(\mathbf{1}-\left(\left(\frac{X}{c}\right)\mathbf{a}b+\left(\frac{1}{r}\right)d\right)\right)'\left(\mathbf{1}-\left(\left(\frac{X}{c}\right)\mathbf{a}b+\left(\frac{1}{r}\right)d\right)\right)$$

$$=n-2\mathbf{1}'\left(\frac{X}{c}\right)\mathbf{a}+\left(\left(\frac{X}{c}\right)\mathbf{a}\right)'\left(\frac{X}{c}\right)\mathbf{a}+n-2\mathbf{1}'\left(\left(\frac{X}{c}\right)\mathbf{a}b+\left(\frac{1}{r}\right)d\right)$$

$$+\left(\left(\frac{X}{c}\right)\mathbf{a}b+\left(\frac{1}{r}\right)d\right)'\left(\left(\frac{X}{c}\right)\mathbf{a}b+\left(\frac{1}{r}\right)d\right)$$

$$=2n-2\mathbf{1}'\left(\frac{X}{c}\right)\mathbf{a}+\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}-2\mathbf{1}'\left(\frac{X}{c}\right)\mathbf{a}b-2\mathbf{1}'\left(\frac{1}{r}\right)d$$

$$+\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}b^2+2\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)bd+\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)d^2 \tag{4.8}$$

Let $\mathbf{0}$ be a zero column (k+1)-vector. We assume throughout the paper that the matrix $\mathbf{X}'\mathbf{X}$ is non-singular. By partially differentiating and equating to a zero vector we have following equations;

$$\frac{\partial}{\partial\mathbf{a}}\varphi(\mathbf{a},b,\text{d})=\mathbf{0}$$

$$\Rightarrow-\left(\frac{X}{c}\right)'\mathbf{1}+\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}-\left(\frac{X}{c}\right)'\mathbf{1}b+\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}b^2+\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)bd=\mathbf{0}, \tag{4.9}$$

$$\frac{\partial}{\partial b}\varphi(\mathbf{a},b,d)=0 \quad ,$$

$$\Rightarrow-\mathbf{1}'\left(\frac{X}{c}\right)\mathbf{a}+\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}b+\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)d=0 \tag{4.10}$$

$$\frac{\partial}{\partial d}\varphi(\mathbf{a},b,d)=0 \quad .$$

$$\Rightarrow-\mathbf{1}'\left(\frac{1}{r}\right)+\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)b+\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)d=0. \tag{4.11}$$

Proposition 1. The sum (and; hence; the mean) of then errors $r\left(\varepsilon_{1c}\right)$/r $=\left(\varepsilon_{1c}\right)$ is zero, i.e.

$$\left(\frac{1}{r}\right)'\left(\mathbf{1}-\left(\frac{c^*}{c}\right)\right)=0 \tag{4.12}$$

Proof of Proposition 1. Let, $\mathbf{0_n}$ be a zero column n-vector (and **0** a zero column vector having dimension $k+1$). By (4.9) and the relation $\left(\frac{c^*}{c}\right)=\left(\frac{X}{c}\right)\mathbf{a}$, we have

$$\Rightarrow -\left(\frac{X}{c}\right)'\mathbf{1}+\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}-\left(\frac{X}{c}\right)'\mathbf{1}b+\left(\frac{X}{c}\right)'\left(\frac{X}{c}\right)\mathbf{a}b^2+\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)bd=\mathbf{0}$$

$$\Rightarrow\left(\frac{X}{c}\right)'\left(-\mathbf{1}+\left(\frac{c^*}{c}\right)-\mathbf{1}b+\left(\frac{c^*}{c}\right)b^2+\left(\frac{1}{r}\right)bd\right)=\mathbf{0}$$

(note here the zero vector has dimension $k+1$)

$$\Rightarrow\left(\frac{X}{c}\right)\left(\frac{X}{c}\right)'\left(-\mathbf{1}+\left(\frac{c^*}{c}\right)-\mathbf{1}b+\left(\frac{c^*}{c}\right)b^2+\left(\frac{1}{r}\right)bd\right)=\left(\frac{X}{c}\right)\left(\frac{X}{c}\right)'\mathbf{0}_n$$

(note here the zero vector has dimension n),

by multiplying to the left by $\left(\left(\frac{X}{c}\right)\left(\frac{X}{c}\right)'\right)^{-1}$, we get

$$\Rightarrow -\mathbf{1}+\left(\frac{c^*}{c}\right)-\mathbf{1}b+\left(\frac{c^*}{c}\right)b^2+\left(\frac{1}{r}\right)bd=\mathbf{0}_n$$

$$\Rightarrow\left(\frac{1}{r}\right)'\left(-\mathbf{1}+\left(\frac{c^*}{c}\right)-\mathbf{1}b+\left(\frac{c^*}{c}\right)b^2+\left(\frac{1}{r}\right)bd\right)=\left(\frac{1}{r}\right)'\mathbf{0}_n$$

$$\Rightarrow\left(\frac{1}{r}\right)'\left(\mathbf{1}-\left(\frac{c^*}{c}\right)\right)=-\left(\frac{1}{r}\right)'\mathbf{1}b+\left(\frac{1}{r}\right)'\left(\frac{c^*}{c}\right)b^2+\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)bd,$$

by noticing that $\left(\frac{c*}{c}\right)=\left(\frac{X}{c}\right)\mathbf{a}$, we have

$$\Rightarrow\left(\frac{1}{r}\right)'\left(\mathbf{1}-\left(\frac{c^*}{c}\right)\right)=-\left(\frac{1}{r}\right)'\mathbf{1}b+\left(\frac{1}{r}\right)'\left(\frac{X}{c}\right)\mathbf{a}b^2+\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)bd$$

Finally, we can note that

$$\Rightarrow -\mathbf{1}'\left(\frac{1}{r}\right)+\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)b+\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)d=0$$

the above relationship is equivalent to (4.7). □

Proposition 2. The sum (and, hence, the mean) of then errors $r\left(\varepsilon_{1r}\right)$/r=$\left(\varepsilon_{1r}\right)$ is zero; i.e.

$$\left(\frac{1}{r}\right)'\left(\mathbf{1}-\left(\frac{r^*}{r}\right)\right)=0 \tag{4.13}$$

Proof of Proposition 2. By (4.11), we have

$$\Rightarrow -\mathbf{1}'\left(\frac{1}{r}\right)+\mathbf{a}'\left(\frac{X}{c}\right)'\left(\frac{1}{r}\right)b+\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)d=0$$

$$\Rightarrow -\left(\frac{1}{r}\right)'\mathbf{1}+\left(\frac{1}{r}\right)'\left(\frac{X}{c}\right)\mathbf{a}b+\left(\frac{1}{r}\right)'\left(\frac{1}{r}\right)d=0$$

$$\Rightarrow \left(\frac{1}{r}\right)'\left(\mathbf{1}-\left(\frac{X}{c}\right)\mathbf{a}b-\left(\frac{1}{r}\right)d\right)=0$$

By $\left(\frac{r^*}{r}\right)=\left(\frac{\mathrm{X}}{\mathrm{c}}\right)\mathbf{a}b+\left(\frac{1}{r}\right)d$, we have our thesis. □

Remark 1. Propositions 1 and 2 imply the equality of the following averages:

$$E\left(\frac{1}{r}\right)=E\left(\frac{c^*/c}{r}\right)$$

and

$$E\left(\frac{1}{r}\right)=E\left(\frac{r^*/r}{r}\right). \tag{4.14}$$

$$E\left(\varepsilon_{1c}\right)=E\left(\frac{1}{r}\right)-E\left(\frac{c^*/c}{r}\right)=0, \tag{4.15}$$

$$E\left(\varepsilon_{1r}\right)=E\left(\frac{1}{r}\right)-E\left(\frac{r^*/r}{r}\right)=0. \tag{4.16}$$

Because **r** is a non-random n x 1 matrix,

$$E\left(r\varepsilon_{1c}\right)=rE\left(\varepsilon_{1c}\right)=r.0=0, \tag{4.17}$$

$$E\left(r\varepsilon_{1r}\right)=rE\left(\varepsilon_{1r}\right)=r.0=0. \tag{4.18}$$

Applicative examples

In this section, two examples with one and several independent variables are shown to illustrate the proposed approach, respectively.

Table 1. Input (crisp)-output (fuzzy) data

Objects i	Visual angle x_i	Subject preference $Y_i = (c_i, r_i)$
1	14.8	(67.7, 42.7)
2	18.0	(69.8, 32.3)
3	22.9	(89.6, 27.1)
4	31.5	(88.5, 26.0)
5	50.3	(71.9, 34.4)
6	126.0	(30.2, 44.8)

Table 2. Chang et al.'s data results

c/c	c*/c	r/r	r*/r	c	c*		r	r*		|c-c*|		|r-r*|	
					ols	lsr		ols	lsr	ols	lsr	ols	lsr
1	1.16	1	1.02	67.7	82.14	78.84	42.7	31.52	43.54	14.44	11.14	11.18	0.84
1	1.11	1	1.09	69.8	80.76	77.50	32.3	31.86	35.09	10.96	7.70	0.44	2.79
1	0.84	1	1.00	89.6	78.65	75.44	27.1	32.37	27.14	10.95	14.16	5.27	0.04
1	0.81	1	1.00	88.5	74.96	71.84	26.0	33.26	26.10	13.54	16.66	7.26	0.10
1	0.89	1	0.92	71.9	66.87	63.95	34.4	35.21	31.88	5.03	7.95	0.81	2.52
1	1.07	1	0.94	30.2	34.32	32.18	44.8	43.08	42.33	4.12	1.98	1.72	2.47
$\frac{1}{6}\sum_1^6 (c/c - c^*/c) \cong 0$		$\frac{1}{6}\sum_1^6 (r/r - r^*/r) \cong 0$							mae =	9.84	9.93	4.45	1.46

Iteration number: 318

Example 1. (*one independent variable*). The data in Table 1 were drawn by Chang et al. (1996). These data refer to the effects of the position of a video display terminal on an operator. The dependent fuzzy variable is the subject preference given by a crisp value, in a scale 0-100, and a corresponding spread, while the (crisp) independent variable is the visual angle of the display [8].

The parameters of the double linear adaptive fuzzy regression model have been estimated as follows by using the proposed LSR approach;

$$\mathbf{1} = \left(\frac{c^*}{c}\right) + \mathbf{r}\varepsilon_{1c}, \quad \left(\frac{c^*}{c}\right) = \left(\frac{\mathrm{X}}{\mathrm{c}}\right)\mathbf{a}, \qquad \textit{core regression model,}$$

$$\mathbf{1} = \left(\frac{r^*}{r}\right) + \mathbf{r}\varepsilon_{1r}, \quad \left(\frac{r^*}{r}\right) = \left(\frac{c^*}{c}\right)b + \left(\frac{1}{r}\right)d, \quad \textit{spread regression model,}$$

and the parameters were found as follows;

$$\mathbf{a} = (85.05, -0.42)',$$
$$b = 0.61,$$
$$d = 13.24.$$

Table 2 indicates Chang et al.'s data results. According to MAE, we can say that the OLS Approach gave a little bit better results than the LSR Approach while estimating center values c_i. Also, we can say that the LSR Approach showed better performance than the OLS Approach while estimating spread values r_i.

Table 3. Input (crisp)-output (fuzzy) data

Objects	Crisp input			Fuzzy output
i	x_{1i}	x_{2i}	x_{3i}	$Y_i = (c_i, r_i)$
1	3	5	9	(96, 42)
2	14	8	3	(120, 47)
3	7	1	4	(52, 33)
4	11	7	3	(106, 45)
5	7	12	15	(189, 79)
6	8	15	10	(194, 65)
7	3	9	6	(107, 42)
8	12	15	11	(216, 78)
9	10	5	8	(108, 52)
10	9	7	4	(103, 44)

One can note that, as expected, the operator's preference rapidly decades, as the visual angle increases [8].

Example 2. (*several independent variables*). The procedure was applied to the data shown in Table 3, drawn by Tanaka (1987).

The parameters were calculated as follows;

$$\mathbf{a} = (1.78,\ 3.11,\ 7.66,\ 5.01)',$$
$$b = 1.02,$$
$$d = -1.14.$$

and the results are summarized below Table 4.

Table 4. Tanaka's data results

c/c	c*/c	r/r	r*/r	c	c*		r	r*		|c-c*|		|r-r*|	
					ols	lsr		ols	lsr	ols	lsr	ols	lsr
1	0.98	1	0.98	96	93.86	94.49	42	42.38	41.04	2.14	1.51	0.38	0.96
1	1.01	1	1.00	120	122.04	121.58	47	50.63	47.44	2.04	1.58	3.63	0.44
1	0.98	1	1.00	52	50.11	51.23	33	29.56	33.12	1.89	0.77	3.44	0.12
1	0.99	1	0.98	106	104.13	104.60	45	45.38	44.16	1.87	1.40	0.38	0.84
1	1.00	1	1.01	189	193.31	190.60	79	71.51	80.15	4.31	1.60	7.49	1.15
1	0.99	1	0.99	194	192.58	191.62	65	71.30	64.37	1.42	2.38	6.30	0.63
1	1.03	1	1.02	107	109.12	110.08	42	46.55	42.95	2.12	3.08	4.55	0.95
1	0.97	1	0.97	216	211.71	209.06	78.0	76.90	75.89	4.29	6.94	1.10	2.11
1	1.03	1	1.03	108	112.48	111.23	52	47.86	53.51	4.48	3.23	4.14	1.51
1	1.00	1	1.00	103	102.67	103.39	44	44.96	43.93	0.33	0.39	0.96	0.07
									mae =	2.49	2.29	3.24	0.88

$\frac{1}{6}\sum_1^6 (c/c - c^*/c) \cong 0$ $\quad \frac{1}{6}\sum_1^6 (r/r - r^*/r) \cong 0$

Iteration number: 366

Table 4 indicates Tanaka.'s data results. According to mae, we can say that the LSR Approach gave better results than the OLS Approach while estimating center values c_i and spread values r_i.

Conclusions and Future Work

In this study, a regression method called Least Squares Ratio (LSR) approach to fuzzy linear regression was explained as an alternative method to other techniques. According to Mean Absolute Error (MAE), we can say that the LSR Approach generally gave better results than the OLS Approach. In addition to this, in the case of the presence of outliers and/or extreme values, the LSR Approach will show better performance than OLS because the LSR Approach is more sensitive to them. For future work, the LSR and OLS Approaches can be compared to which technique give better results by using different error criteria. Also, their performance can be explored in the case of different fuzzy numbers such as non-symmetric triangular and trapezoidal fuzzy numbers etc.

Acknowledgements

I would like to thank JForce Information Technologies Inc. Istanbul, Turkey for supporting this study.

References

1. B. Heshmaty, A. Kandel, Fuzzy linear regression and its applications to forecasting in uncertain environment, Fuzzy Sets Systems 15:2 (1985) 159-191.
2. C. Cheng, E.S. Lee, Fuzzy Regression with Radial Basis Function Network, Fuzzy Sets and Systems 119:2 (2001) 291-301.
3. C. Kao, C.L. Chyu, Least-Squares Estimates in Fuzzy Regression Analysis, European Journal of Operational Research 148 (2003) 426-43.
4. D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
5. H. Tanaka, Fuzzy data analysis by possibilistic linear models, Fuzzy Sets Systems 24 (1987) 363-375.
6. H. Tanaka, S. Uejima, K. Asai, Linear regression analysis with fuzzy model, IEEE Trans. Systems Man Cybernet 12:6 (1982) 903-907.
7. H. Tanaka, J. Watada, Possibilistic linear systems and their application to the linear regression model, Fuzzy Sets Systems 27 (1988) 145-160.
8. H. Tanaka, H. Ishibuchi, Identification of possibilistic linear systems by quadratic membership functions of fuzzy parameters, Fuzzy Sets Systems 41 (1991) 145-160.
9. H. Tanaka, H. Ishibuchi, S. Yoshikawa, Exponential possibility regression analysis, Fuzzy Sets Systems 69 (1995) 305-318.
10. H.J. Zimmermann, Fuzzy Set Theory-and Its Application, Kluwer Academic Press, Dordrecht, 1991.
11. J.J. Buckley, Fuzzy Probability and Statistics, Springer, 2006.
12. O. Akbilgic, E.D. Akinci, A Novel Regression Approach: Least Squares Ratio, Communications in Statistics - Theory and Methods 38:9 (2009) 1539-1545.
13. P. D'Urso, T. Gastaldi, A Least-Squares Approach to Fuzzy Linear Regression Analysis, Computational Statistics & Data Analysis 34 (2000) 427-440.
14. P.T. Chang, E.S. Lee, S.A. Konz, Applying fuzzy linear regression to VDT legibility, Fuzzy Sets Systems 80:2 (1996) 197-204.
15. W. Xizhao, H. Minghu, Fuzzy linear regression analysis, Fuzzy Sets Systems 51 (1992) 179-188.